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"Adversarial Coordination and Public Information Design"
Online Appendix

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Abstract

This document contains expanded proofs (with additional algebraic derivations) for some of the results in the manuscript "Adversarial Coordination and Public Information Design." All numbered items contain the prefix "S." Any numbered reference without the prefix "S" refers to an item in the main text. Section S1 contains the expanded proof of Theorem 3 in the main text, accommodating for (a) a richer family of economies in which payoffs depend on the size of the aggregate pledge even after conditioning on the regime outcome and (b) the possibility that the prior, as well as the distribution from which the agents' private signals are drawn, have bounded support. Section S2 contains the expanded proof of Example 2 in the main text, whereas Section S3 the expanded proof of Example 3. Finally, Section S4 discusses how the results in the baseline model extend to economies with richer payoff specifications and in which the regime outcome also depends on variables not directly observable by the policy maker.

Section S1: Extended Proof of Theorem 3

Here we show that Theorem 3 in the main body extends to a richer family of economies in which payoffs depend on the size of the aggregate pledge A beyond the effect through the determination of the regime outcome. That is,

$$u(\theta, A) \equiv g(\theta, A)\mathbf{1}\left\{R(\theta, A) > 0\right\} + b(\theta, A)\mathbf{1}\left\{R(\theta, A) \leq 0\right\},$$

with $g(\theta, A) > 0 > b(\theta, A)$ for all (θ, A). The proof also accommodates for the possibility that the prior, as well as the distribution from which the agents' private signals are drawn, have bounded support.

Condition M-generalized. *The following properties hold*:

1. $\inf \Theta(\bar{x}_G) \leq 0$;
2. *(2') $|u(\theta, 1 - P(x|\theta))|$ is log-supermodular over*[1]

$$\left\{(\theta, x) \in [0,1] \times \mathbb{R} : u(\theta, 1 - P(x|\theta)) \leq 0\right\},$$

and, (2") for any $\theta_0, \theta_1 \in [0,1]$, with $\theta_0 < \theta_1$, and $x \leq \bar{x}_G$ such that (a) $u(\theta_1, 1 - P(x|\theta_1)) \leq 0$ and (b) $x \in \varrho_{\theta_0}$,

$$\frac{U^P(\theta_1, 1) - U^P(\theta_1, 0)}{U^P(\theta_0, 1) - U^P(\theta_0, 0)} > \frac{p\left(x|\theta_1\right) u\left(\theta_1, 1 - P\left(x|\theta_1\right)\right)}{p\left(x|\theta_0\right) u\left(\theta_0, 1 - P\left(x|\theta_0\right)\right)}. \tag{1}$$

Part (2') of Property 2 says that the percentage reduction in the investors' loss from pledging when default occurs due to higher fundamentals is larger when more investors pledge. Precisely, for any $\theta' < \theta''$ and $x' < x''$ such that $u(\theta'', 1 - P(x'|\theta'')) < 0$,

$$\frac{u(\theta', 1 - P(x''|\theta')) - u(\theta'', 1 - P(x''|\theta''))}{u(\theta', 1 - P(x''|\theta'))} \leq \frac{u(\theta', 1 - P(x'|\theta')) - u(\theta'', 1 - P(x'|\theta''))}{u(\theta', 1 - P(x'|\theta'))}. \tag{2}$$

Note that $u(\theta'', 1 - P(x'|\theta'')) < 0$ implies that $u(\theta', 1 - P(x'|\theta')), u(\theta', 1 - P(x''|\theta')), u(\theta'', 1 - P(x''|\theta'')) < 0$. The left-hand side of (2) is thus the percentage reduction in the loss from

[1] The log-supermodularity of $|u(\theta, 1 - P(x|\theta))|$ means that, for any $x', x'' \in \mathbb{R}$, with $x' < x''$, and any $\theta', \theta'' \in \Theta$, with $\theta'' > \theta'$, such that $u(\theta'', 1 - P(x'|\theta'')) \leq 0$,

$$u(\theta'', 1 - P(x''|\theta''))u(\theta', 1 - P(x'|\theta')) \geq u(\theta'', 1 - P(x'|\theta''))u(\theta', 1 - P(x''|\theta')).$$

pledging under regime change (i.e., default) when fundamentals improve from θ' to θ'' and investors pledge when, and only when, they receive signal $x \geq x''$. The right-hand side of (2), instead, is the analogous reduction when investors are more lenient and pledge when, and only when, they receive signals $x \geq x'$, with $x' < x''$. Importantly, this property is required to hold only for fundamentals θ and signal thresholds x for which the investors' expected payoff from pledging, $u(\theta, 1 - P(x|\theta))$, is non-positive. Also note that this property trivially holds in the baseline model, because payoffs $u(\theta, A)$ are invariant in A conditional on the regime outcome.

Part (2") of Condition M says that the value the policy maker assigns to avoiding regime change is significantly larger when fundamentals are stronger. It says that the policy maker's preferences for saving institutions with stronger fundamentals are sufficiently strong. Specifically, it identifies the critical strength of such a preference necessary to compensate for the possibility that non-monotone rules may permit the policy maker to spare regime change over larger measure of fundamentals. The benefit that the policy maker derives from changing the investors' behavior (inducing all investors to pledge starting from a situation in which no investor pledges) must increase with the fundamentals at a sufficiently high rate, with the critical rate determined by a combination of the investors' payoffs and beliefs (the right-hand-side of (1)).

Proof of Theorem 3.

Without loss of generality, assume that the policy $\Gamma = (\mathcal{S}, \pi)$ (a) is a (possibly stochastic) "pass/fail"policy (i.e., $\mathcal{S} = \{0, 1\}$, with $\pi(1|\theta) = 1 - \pi(0|\theta)$ denoting the probability that signal $s = 1$ is disclosed when the fundamentals are θ), (b) is such that $\pi(1|\theta) = 0$ for all $\theta \leq 0$ and $\pi(1|\theta) = 1$ for all $\theta > 1$, and (c) satisfies the perfect-coordination property (PCP). Theorems 1 and 2 imply that, if Γ does not satisfy these properties, there exists another policy Γ' that satisfies these properties and yields the policy maker a payoff weakly higher than Γ. The proof then follows from applying the arguments below to Γ' instead of Γ.

Suppose that Γ is such that there exists no $\hat{\theta}$ such that $\pi(1|\theta) = 0$ for F-almost all $\theta \leq \hat{\theta}$ and $\pi(1|\theta) = 1$ for F-almost all $\theta > \hat{\theta}$.[2] We establish the result by showing that there exists a deterministic monotone policy $\Gamma^{\hat{\theta}} = (\{0, 1\}, \pi^{\hat{\theta}})$ satisfying PCP that yields the policy maker a payoff strictly higher than Γ.

For the policy Γ to satisfy PCP, it must be that, when the policy discloses the signal $s = 1$, $U^{\Gamma}(x, 1|x) > 0$ for all x such that $(x, 1)$ are mutually consistent, where $U^{\Gamma}(x, 1|x)$ is the expected payoff of an investor with signal x who hears that $s = 1$ and who expects all other investors to follow a cut-off policy with cut-off x.

[2]If this not the case, then the deterministic monotone policy $\Gamma^{\hat{\theta}} = (\{0, 1\}, \pi^{\hat{\theta}})$ with cut-off $\hat{\theta}$ also satisfies PCP and yields the policy maker the same payoff as Γ, in which case the result trivially holds.

Let $\mathbb{G}$ denote the set of policies $\Gamma' = (\mathcal{S}, \pi')$ that, in addition to properties (a) and (b) above, are such that $U^{\Gamma'}(x, 1|x) \geq 0$ for all x such that $(x, 1)$ are mutually consistent. [3] For any $\Gamma \in \mathbb{G}$, let $\mathcal{U}^P[\Gamma]$ denote the policy maker's ex-ante expected payoff when, under Γ, investors pledge after hearing that $s = 1$ and refrain from pledging after hearing that $s = 0$. Denote by $\arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\}$ the set of policies that maximize the policy maker's payoff over $\mathbb{G}$.[4]

Step 1 below shows that any $\Gamma \in \arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\}$ is such that $\pi(1|\theta) = 0$ for F-almost all $\theta \leq \theta^*$ and $\pi(1|\theta) = 1$ for F-almost all $\theta > \theta^*$, with θ^* as defined in (3) in the main text. Step 2 then shows that the policy maker's payoff under the optimal deterministic monotone policy $\Gamma^{\theta^*} = (\{0, 1\}, \pi^{\theta^*})$ with cut-off θ^* can be approximated arbitrarily well by a deterministic monotone policy $\Gamma^{\hat{\theta}} = (\{0, 1\}, \pi^{\hat{\theta}}) \in \mathbb{G}$ that satisfies PCP, thus establishing the theorem.

Step 1. Given any policy Γ, let

$$X^{\Gamma} \equiv \{x : (x, 1) \text{ } \Gamma\text{-mutually consistent and } U^{\Gamma}(x, 1|x) = 0\}.$$

Take any policy $\Gamma' \in \mathbb{G}$ for which there exists no $\hat{\theta}$ such that $\pi'(1|\theta) = 0$ for F-almost all $\theta \leq \hat{\theta}$ and $\pi'(1|\theta) = 1$ for F-almost all $\theta > \hat{\theta}$. Clearly, if $X^{\Gamma'} = \emptyset$, there exists another policy $\Gamma'' \in \mathbb{G}$ that yields the policy maker a payoff strictly higher than Γ'.[5] Thus, assume that $X^{\Gamma'} \neq \emptyset$, and let $\bar{x} \equiv \sup X^{\Gamma'}$. Claim S1-A below shows that the set $\{\theta \in \Theta(\bar{x}) : \pi'(1|\theta) < 1\}$ has strict positive F-measure. Claim S1-B shows that, given any $\Gamma' \in \mathbb{G}$ for which the posterior beliefs of the marginal investor with signal $\bar{x}$ differ from those obtained by Bayes rule conditioning on the event that fundamentals are above some threshold $\hat{\theta}$, there exists another policy $\Gamma'' \in \mathbb{G}$ that yields the policy maker a payoff strictly higher than Γ'. Finally, Claim S1-C shows that, under the properties in Condition M, the only policies $\Gamma' \in \mathbb{G}$ that generate posterior beliefs for the marginal investors with signal $\bar{x}$ equal to those obtained from Bayes rule by conditioning on the event that fundamentals are above some threshold $\hat{\theta}$ are such that $\pi'(1|\theta) = 0$ for F-almost all $\theta \leq \theta^*$ and $\pi'(1|\theta) = 1$ for F-almost all $\theta > \theta^*$. Jointly, the three claims thus establish the result that any policy $\Gamma \in \arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\}$, is such that $\pi(1|\theta) = 0$ for F-almost all $\theta \leq \theta^*$ and $\pi(1|\theta) = 1$ for F-almost all $\theta > \theta^*$.

[3] As explained in the main text, some policies Γ' in $\mathbb{G}$ need not satisfy PCP, namely those for which there exists x, with $(x, 1)$ mutually consistent, such that $U^{\Gamma'}(x, 1|x) = 0$.

[4] That $\arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\} \neq \emptyset$ follows from the compactness of $\mathbb{G}$ and the upper hemi-continuity of $\mathcal{U}^P$.

[5] In fact, because there exists no $\hat{\theta}$ such that $\pi'(1|\theta) = 0$ for F-almost all $\theta \leq \hat{\theta}$ and $\pi'(1|\theta) = 1$ for F-almost all $\theta > \hat{\theta}$, there must exists a set $(\theta', \theta'') \subseteq [0, 1]$ of F-positive measure over which $\pi'(1|\theta) < 1$. The policy Γ'' can then be obtained from Γ' by increasing $\pi'(1|\theta)$ over such a set. Provided the increase is small, $\Gamma'' \in \mathbb{G}$. Because $U^P(\theta, 1) > U^P(\theta, 0)$ over $[0, 1]$, the policy maker's payoff under Γ'' is strictly higher than under Γ'.

Given any x, let $\theta_0(x)$ be the fundamental threshold below which the investors' expected payoff differential is negative and above which it is positive, when all investors follow a cut-off strategy with cut-off x.[6] For any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}$, let $p^{\Gamma}(x,1) \equiv \int_{-\infty}^{+\infty} \pi(1|\theta)p(x|\theta)\mathrm{d}F(\theta)$ denote the joint probability density of the exogenous signal x and the endogenous signal $s = 1$.

Claim S1-A. *For any $\Gamma' = (\{0,1\}, \pi') \in \mathbb{G}$ such that $X^{\Gamma'} \neq \emptyset$, $\{\theta \in \Theta(\bar{x}) : \pi'(1|\theta) < 1\}$ has strict positive F-measure.*

Proof of Claim S1-A. Suppose, by contradiction, that $\pi'(1|\theta) = 1$ for F-almost all $\theta \in \Theta(\bar{x})$. Property 1 in Condition M then implies that $\bar{x} > \bar{x}_G$, where

$$\bar{x}_G \equiv \sup\{x \in \mathbb{R} : \int_{\Theta} u(\theta, 1 - P(x|\theta)) \mathbf{1}(\theta \geq 0) p(x|\theta) \mathrm{d}F(\theta) \leq 0\}. \tag{S1}$$

In fact, if this was not the case, the monotonicity of $\Theta(\cdot)$ would imply that $\inf \Theta(\bar{x}) \leq \inf \Theta(\bar{x}_G) < 0$. That $\pi'(1|\theta) = 1$ for F-almost all $\theta \in \Theta(\bar{x})$ would then imply that $\pi'(1|\theta) = 1$ for a set of fundamentals $\theta < 0$ of strict positive F-measure, which is inconsistent with the assumption that $\Gamma' \in \mathbb{G}$. Thus, necessarily, $\bar{x} > \bar{x}_G$.

Now suppose that $\inf \Theta(\bar{x}) \geq 0$. That $\pi'(1|\theta) = 1$ for F-almost all $\theta \in \Theta(\bar{x})$ means that, from the perspective of an investor with signal $\bar{x}$, the information conveyed by the announcement that $s = 1$ under Γ' is the same as under the monotone deterministic policy $\Gamma^0 = (\{0,1\}, \pi^0)$ with cut-off $\hat{\theta} = 0$. As a result, $U^{\Gamma'}(\bar{x}, 1|\bar{x}) = U^{\Gamma^0}(\bar{x}, 1|\bar{x})$. Because $\bar{x} > \bar{x}_G$, and because, by definition of $\bar{x}_G$, $U^{\Gamma^0}(x, 1|x) > 0$ for all $x > \bar{x}_G$, it must be that $U^{\Gamma'}(\bar{x}, 1|\bar{x}) > 0$, which contradicts the assumption that $\bar{x} \in X^{\Gamma'}$. Hence, it must be that $\inf \Theta(\bar{x}) < 0$. As explained above, however, this is inconsistent with the assumption that $\Gamma' \in \mathbb{G}$. $\square$

Next, for any $\Gamma' = (\{0,1\}, \pi') \in \mathbb{G}$, let

$$\theta_H \equiv \sup\{\theta \in \Theta : \exists \delta > 0 \, \text{s.t.} \, \pi'(1|\theta') < 1 \, \text{for} \, F\text{-almost all} \, \theta' \in [\theta - \delta, \theta)\}.$$

The result in Claim S1-A above implies that θ_H is such that $\theta_H > \inf \Theta(\bar{x})$.

Claim S1-B. *Take any $\Gamma' = (\{0,1\}, \pi') \in \mathbb{G}$ such that $X^{\Gamma'} \neq \emptyset$. Suppose that*

$$\{\theta \in (\inf \Theta(\bar{x}), \theta_H) : \pi'(1|\theta) > 0\} \text{ has strict positive } F\text{-measure.} \tag{S2}$$

Then, there exists another policy $\Gamma'' \in \mathbb{G}$ that yields the policy maker a strictly higher payoff.

Proof of Claim S1-B. The proof below distinguishes two cases.

[6] When the default outcome is a function of A and θ only, as in the baseline model of Section 2 in the main text, $\theta_0(x)$ coincides with the threshold below which default occurs and above which it does not occur, when agents follow a cut-off strategy with cut-off x.

Case 1: $\inf \Theta(\bar{x}) < \theta_0(\bar{x}) < \theta_H$. Consider the policy $\Gamma^{\epsilon,\delta} = (\{0,1\}, \pi^{\epsilon,\delta})$ defined by $\pi^{\epsilon,\delta}(1|\theta) = \pi'(1|\theta)$ for all $\theta \leq \theta_0(\bar{x}+\delta)$, with $\delta > 0$ small so that $\theta_0(\bar{x}+\delta) < \theta_H$, and $\pi^{\epsilon,\delta}(1|\theta) = \min\{\pi'(1|\theta)+\epsilon, 1\})$ for all $\theta > \theta_0(\bar{x}+\delta)$, with $\epsilon > 0$ also small. To see that, when ϵ and δ are small, $\Gamma^{\epsilon,\delta} \in \mathbb{G}$, note that, by definition of $\theta_0(\cdot)$, for any x, and any $\theta > \theta_0(x)$, $u(\theta, 1-P(x|\theta)) > 0$. This property, together with the monotonicity of $\theta_0(\cdot)$, jointly imply that, for any $x \leq \bar{x}+\delta$,

$$\int_{-\infty}^{\infty} u(\theta, 1-P(x|\theta)) \left[\pi'(1|\theta)\mathbf{1}\{\theta \leq \theta_0(\bar{x}+\delta)\} + \min\{\pi'(1|\theta)+\epsilon, 1\}\mathbf{1}\{\theta > \theta_0(\bar{x}+\delta)\}\right] p(x|\theta)\mathrm{d}F(\theta)$$
$$\geq \int_{-\infty}^{\infty} u(\theta, 1-P(x|\theta))\pi'(1|\theta)p(x|\theta)\mathrm{d}F(\theta). \tag{S3}$$

The inequality follows from the fact that, when $x \leq \bar{x}+\delta$, $u(\theta, 1-P(x|\theta)) > 0$ for any $\theta > \theta_0(\bar{x}+\delta)$. Because $\Gamma' \in \mathbb{G}$, the right-hand side of (S3) is non-negative.[7] Hence, for any $x \leq \bar{x}+\delta$ such that $(x,1)$ are mutually consistent under $\Gamma^{\epsilon,\delta}$, because the left-hand side of (S3) is equal to $U^{\Gamma^{\epsilon,\delta}}(x,1|x)p^{\Gamma^{\epsilon,\delta}}(x,1)$ and because, for such x, $p^{\Gamma^{\epsilon,\delta}}(x,1) > 0$, we have that $U^{\Gamma^{\epsilon,\delta}}(x,1|x) \geq 0$. That $U^{\Gamma^{\epsilon,\delta}}(x,1|x) \geq 0$ also for all $x > \bar{x}+\delta$ such that $(x,1)$ are mutually consistent under $\Gamma^{\epsilon,\delta}$ follows from the fact that, by definition of $\bar{x}$, for any $x \geq \bar{x}+\delta$, the function $J(x) \equiv \int_{-\infty}^{+\infty} u(\theta, 1-P(x|\theta))\pi'(1|\theta)p(x|\theta)\mathrm{d}F(\theta)$ is bounded away from 0, along with the fact that, for any $\delta > 0$, the function family $\left(J^{\epsilon,\delta}(\cdot)\right)_\epsilon$ whose elements $J^{\epsilon,\delta}(\cdot)$ are given by $J^{\epsilon,\delta}(x) \equiv \int_{-\infty}^{+\infty} u(\theta, 1-P(x|\theta))\pi^{\epsilon,\delta}(1|\theta)p(x|\theta)\mathrm{d}F(\theta)$, is continuous in ϵ in the sup-norm in a neighborhood of 0.[8] Because the new policy $\Gamma^{\epsilon,\delta} \in \mathbb{G}$ is such that $\pi^{\epsilon,\delta}(1|\theta) \geq \pi'(1|\theta)$ for all θ, with the inequality strict over a set of fundamentals $\theta < 1$ of F-positive measure, the policy maker's payoff under $\Gamma^{\epsilon,\delta}$ is strictly higher than under Γ', as claimed.

Case 2: $\inf \Theta(\bar{x}) < \theta_H \leq \theta_0(\bar{x})$. Consider the monotone deterministic policy $\Gamma^0 = \{\{0,1\}, \pi^0\}$ with cut-off $\hat{\theta} = 0$. Then, for any $x \geq \bar{x}$,

$$\int_0^{+\infty} u(\theta, 1-P(x|\theta))\pi'(1|\theta)p(x|\theta)\mathrm{d}F(\theta) \geq \int_0^{+\infty} u(\theta, 1-P(x|\theta))p(x|\theta)\mathrm{d}F(\theta),$$

where the inequality follows from (i) the fact that, for any $x \geq \bar{x}$ and any $\theta \leq \theta_0(\bar{x})$, $u(\theta, 1-P(x|\theta)) < 0$, along with (ii) the fact that $\pi'(1|\theta) = 1$ for F-almost all $\theta \geq \theta_0(x) \geq \theta_0(\bar{x}) \geq \theta_H$. Furthermore, when $x = \bar{x}$, the above inequality is strict and, because $p^{\Gamma^0}(\bar{x},1) > p^{\Gamma'}(\bar{x},1) > 0$, it implies that $U^{\Gamma^0}(\bar{x},1|\bar{x}) < U^{\Gamma'}(\bar{x},1|\bar{x}) = 0$. By continuity of $U^{\Gamma^0}(x,1|x)$ in x, we thus have that $\bar{x} < \bar{x}_G$. This property in turn permits us to apply part (2") of Condition M to $\bar{x}$ in the arguments below.

[7]To see this, note that either $(x,1)$ are not mutually consistent under Γ', in which case the right-hand side of (S3) is zero, or they are mutually consistent, in which case the right-hand side of (S3) is equal to $U^{\Gamma'}(x,1|x)p^{\Gamma'}(x,1)$, which is non-negative because $p^{\Gamma'}(x,1) > 0$ and $U^{\Gamma'}(x,1|x) \geq 0$.

[8]That is, $\forall z > 0$, $\exists \Delta > 0$ so that $\forall\, 0 < \epsilon < \Delta$, $|J^{\epsilon,\delta}(x) - J(x)| \leq z$, $\forall x \geq \bar{x}+\delta$.

Next, let $\theta_L \equiv \inf\{\theta \in \Theta : \exists \delta > 0 \,\text{s.t.}\, \pi'(1|\theta') > 0,\ F\text{-almost all}\, \theta' \in [\theta, \theta + \delta)\}$. That $\theta_L < \theta_H$ follows from the assumption that $\{\theta \in (\inf \Theta(\bar{x}), \theta_H) : \pi'(1|\theta) > 0\}$ has strict positive F-measure. Furthermore, $u(\theta_L, 1 - P(\bar{x}|\theta_L)) < 0$.[9] Also observe that $\inf \Theta(\bar{x}) < \theta_L$. This follows from the fact that, as shown above, $\bar{x} < \bar{x}_G$, which, together with Property 1 in Condition M, implies that $\inf \Theta(\bar{x}) < 0$. Because $\theta_L \geq 0$, we thus have that $\inf \Theta(\bar{x}) < \theta_L$.

For any $\gamma > 0$, let $\theta_L^\gamma \equiv \theta_L + \gamma$ and $\theta_H^\gamma \equiv \theta_H - \gamma$. Pick $\gamma, e_L, e_H > 0$ small such that (i) $\pi'(1|\theta_L^\gamma) > 0$ and $\pi'(1|\theta) > 0$ for $F-$almost $\theta \in (\theta_L^\gamma, \theta_L^\gamma + e_L)$, (ii) $\pi'(1|\theta_H^\gamma) < 1$ and $\pi'(1|\theta) < 1$ for $F-$almost all $\theta \in (\theta_H^\gamma - e_H, \theta_H^\gamma)$, and (iii) $\theta_L^\gamma + e_L < \theta_H^\gamma - e_H$.[10] Next, pick $\eta \in (0, \bar{x}_G - \bar{x})$ small such that $U^{\Gamma'}(x, 1|x) > \eta$ for all $x \geq \bar{x} + \eta$. Pick $\epsilon > 0$ also small and let $\delta(\epsilon, \eta)$ be implicitly defined by

$$\begin{gathered}\int_{\theta_L^\gamma}^{\theta_L^\gamma+\epsilon} u(\theta, 1 - P(\bar{x} + \eta|\theta))\pi'(1|\theta)p(\bar{x} + \eta|\theta)\mathrm{d}F(\theta) = \\ \int_{\theta_H^\gamma-\delta(\epsilon,\eta)}^{\theta_H^\gamma} u(\theta, 1 - P(\bar{x} + \eta|\theta))(1 - \pi'(1|\theta))p(\bar{x} + \eta|\theta)\mathrm{d}F(\theta).\end{gathered} \tag{S4}$$

For $\epsilon > 0$ small, $\theta_L^\gamma + \epsilon < \theta_H^\gamma - \delta(\varepsilon, \eta)$. Consider the policy $\Gamma^{\epsilon,\gamma,\eta} = (\{0, 1\}, \pi^{\epsilon,\gamma,\eta})$ defined by the following properties: (a) $\pi^{\epsilon,\gamma,\eta}(1|\theta) = \pi'(1|\theta)$ for all $\theta \notin \{[\theta_L^\gamma, \theta_L^\gamma + \epsilon] \cup [\theta_H^\gamma - \delta(\epsilon, \eta), \theta_H^\gamma]\}$; (b) $\pi^{\epsilon,\gamma,\eta}(1|\theta) = 0$ for all $\theta \in [\theta_L^\gamma, \theta_L^\gamma + \epsilon]$; and (c) $\pi^{\epsilon,\gamma,\eta}(1|\theta) = 1$ for all $\theta \in [\theta_H^\gamma - \delta(\epsilon, \eta), \theta_H^\gamma]$. Note that Condition (S4) implies that $U^{\Gamma^{\epsilon,\gamma,\eta}}(\bar{x} + \eta, 1|\bar{x} + \eta) = U^{\Gamma'}(\bar{x} + \eta, 1|\bar{x} + \eta) > 0$.

We now show that $U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0$ for any x with $(x, 1)$ mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$. Clearly, for any (ϵ, γ, η), and any $x \leq x^*(\theta_L)$ such that $(x, 1)$ are mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$ (alternatively, under Γ') $U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) > 0$ (alternatively, $U^{\Gamma'}(x, 1|x) > 0$). This is because, for any such x, $\theta_0(x) < \theta_L$ implying that $u(\theta, 1 - P(x|\theta)) > 0$ for all $\theta > \theta_L$. The result then follows from the fact that, under both Γ' and $\Gamma^{\epsilon,\gamma,\eta}$, $\int_{-\infty}^{\theta_L} \pi^{\epsilon,\gamma,\eta}(1|\theta)\,\mathrm{d}F(\theta) = \int_{-\infty}^{\theta_L} \pi'(1|\theta)\,\mathrm{d}F(\theta) = 0$, meaning that all agents assign probability one to the event that $\theta \geq \theta_L$.

Furthermore, the continuity of $\int_{\theta_L}^{+\infty} u(\theta, 1 - P(x|\theta))\, p(x|\theta)\, \pi'(1|\theta)\,\mathrm{d}F(\theta)$ in x, along with the fact that $\int_{\theta_L}^{+\infty} u(\theta, 1 - P(x|\theta))\, p(x|\theta)\, \pi'(1|\theta)\,\mathrm{d}F(\theta) > \eta$ for all $x \geq \bar{x}+\eta$ with $(x, 1)$ mutually consistent under Γ', imply that there exists $\xi > 0$ so that, for any $x \in [x^*(\theta_L), x^*(\theta_L) + \xi] \cup [\bar{x} + \eta, +\infty)$, if $(x, 1)$ are mutually consistent under Γ', then $U^{\Gamma'}(x, 1|x)p^{\Gamma'}(x, 1) > \xi$.

Let $S^{\Gamma^{\epsilon,\gamma,\eta}}(x) \equiv \int_{\theta_L}^{+\infty} u(\theta, 1 - P(x|\theta))\, p(x|\theta)\, \pi^{\epsilon,\gamma,\eta}(1|\theta)\,\mathrm{d}F(\theta)$. For any η, the function family $\left(S^{\Gamma^{\epsilon,\gamma,\eta}}(\cdot)\right)_{\epsilon,\gamma}$ is continuous in (γ, ϵ) in the sup-norm, in a neighborhood of $(0, 0)$[11] and $x^*(\theta)$ is continuous in θ. Hence, there exist $\bar{\gamma}, \bar{\epsilon} > 0$ such that, when $\gamma \leq \bar{\gamma}$ and $\epsilon \leq \bar{\epsilon}$, for any $x \notin (x^*(\theta_L^\gamma + \epsilon), \bar{x} + \eta)$ such that $(x, 1)$ are mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$, $U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0$.

[9] That $u(\theta_L, 1 - P(\bar{x}|\theta_L)) < 0$ follows from the fact that, by definition of $\bar{x}$ and θ_L, $\int_{\theta_L}^{+\infty} u(\theta, 1 - P(\bar{x}|\theta))\pi'(1|\theta)p(\bar{x}|\theta)dF(\theta) = 0$, together with the single-crossing property of $u(\theta, 1 - P(\bar{x}|\theta))$ in θ.

[10] If a single γ satisfying properties (i)-(iii) does not exist, let $\gamma = (\gamma_L, \gamma_H) \in \mathbb{R}^2_{++}$ satisfying properties (i)-(iii). The arguments below then apply verbatim by letting $\theta_L^\gamma = \theta_L + \gamma_L$ and $\theta_H^\gamma = \theta_H + \gamma_H$.

[11] This means that, for any $z > 0$, there exists $\Delta > 0$ such that, for any (ϵ, γ) with $0 < \epsilon < \Delta$ and $0 < \gamma < \Delta$, and all x, $|S^{\Gamma^{\epsilon,\gamma,\eta}}(x) - S^{\Gamma^{0,0,\eta}}(x)| \leq z$, where $\Gamma^{0,0,\eta} = \Gamma'$.

Next observe that, for any $x \in (x^*(\theta_L^\gamma + \epsilon), x^*(\theta_H^\gamma - \delta(\epsilon, \eta))]$,

$$\begin{aligned} &- \int_{\theta_L^\gamma}^{\theta_L^\gamma + \epsilon} u(\theta, 1 - P(x|\theta)) p(x|\theta) \pi'(1|\theta) \mathrm{d}F(\theta) \\ &+ \int_{\theta_H^\gamma - \delta(\epsilon,\eta)}^{\theta_H^\gamma} u(\theta, 1 - P(x|\theta)) p(x|\theta) (1 - \pi'(1|\theta)) \mathrm{d}F(\theta) \geq 0, \end{aligned}$$

where the inequality follows from the fact that the integrand in the first integral is non-positive, whereas the integrand in the second integral is non-negative. Hence, for any such x, if $(x, 1)$ are mutually consistent under Γ', meaning that $p^{\Gamma'}(x, 1) = \int_{\theta_L}^{+\infty} p(x|\theta) \pi'(1|\theta) \mathrm{d}F(\theta) > 0$, and are also mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$, meaning that $p^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1) = \int_{\theta_L}^{+\infty} p(x|\theta) \pi^{\epsilon,\gamma,\eta}(1|\theta) \mathrm{d}F(\theta) > 0$, it must be that $U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0$. Indeed, for any such x,

$$\begin{aligned} U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) p^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1) = U^{\Gamma'}(x, 1|x) p^{\Gamma'}(x, 1) &- \int_{\theta_L^\gamma}^{\theta_L^\gamma + \epsilon} u(\theta, 1 - P(x|\theta)) p(x|\theta) \pi'(1|\theta) \mathrm{d}F(\theta) \\ &+ \int_{\theta_H^\gamma - \delta(\epsilon,\eta)}^{\theta_H^\gamma} u(\theta, 1 - P(x|\theta)) p(x|\theta) (1 - \pi'(1|\theta)) \mathrm{d}F(\theta) \end{aligned}$$

and $U^{\Gamma'}(x, 1|x) p^{\Gamma'}(x, 1) = \int_{\theta_L}^{+\infty} u(\theta, 1 - P(x|\theta)) p(x|\theta) \pi'(1|\theta) \mathrm{d}F(\theta) \geq 0$. If, instead, for any such x, $(x, 1)$ are mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$ but not under Γ', then

$$\begin{aligned} U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) p^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1) &= \int_{\theta_L}^{+\infty} u(\theta, 1 - P(x|\theta)) p(x|\theta) \pi^{\epsilon,\gamma,\eta}(1|\theta) \mathrm{d}F(\theta) \\ = \int_{\theta_L}^{+\infty} u(\theta, 1 - P(x|\theta)) p(x|\theta) \pi'(1|\theta) \mathrm{d}F(\theta) &- \int_{\theta_L^\gamma}^{\theta_L^\gamma + \epsilon} u(\theta, 1 - P(x|\theta)) p(x|\theta) \pi'(1|\theta) \mathrm{d}F(\theta) \\ &+ \int_{\theta_H^\gamma - \delta(\epsilon,\eta)}^{\theta_H^\gamma} u(\theta, 1 - P(x|\theta)) p(x|\theta) (1 - \pi'(1|\theta)) \mathrm{d}F(\theta) \\ &= \int_{\theta_H^\gamma - \delta(\epsilon,\eta)}^{\theta_H^\gamma} u(\theta, 1 - P(x|\theta)) p(x|\theta) (1 - \pi'(1|\theta)) \mathrm{d}F(\theta) \geq 0, \end{aligned}$$

where the first equality follows from the fact $p^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1) > 0$ and the definition of $U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x)$, the second equality is by construction, the third equality follows from the fact that $p^{\Gamma'}(x, 1) = 0$, and the last inequality follows from the fact that, when $x \in (x^*(\theta_L^\gamma + \epsilon), x^*(\theta_H^\gamma - \delta(\epsilon, \eta))]$, the integrand is non-negative. We thus conclude that, for any such x, $U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0$.

Next, consider $x \in (x^*(\theta_H^\gamma - \delta(\epsilon, \eta)), x^*(\theta_H^\gamma))$. For any (x, θ), let

$$\Delta S(x) \equiv \int_{\theta_L}^{+\infty} u(\tilde{\theta}, 1 - P(x|\tilde{\theta})) p(x|\tilde{\theta}) (\pi^{\epsilon,\gamma,\eta}(1|\tilde{\theta}) - \pi'(1|\tilde{\theta})) \mathrm{d}F(\tilde{\theta})$$

and $q(\theta,x) \equiv |u(\theta,1-P(x|\theta))|\, p(x|\theta)$. Note that, for any $x \in (x^*(\theta_H^\gamma - \delta(\epsilon,\eta)), x^*(\theta_H^\gamma))$,

$$
\begin{aligned}
\Delta S(x) &= \int_{\theta_L^\gamma}^{\theta_H^\gamma-\delta(\epsilon,\eta)} -u(\theta,1-P(x|\theta))\, p(x|\theta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&\quad + \int_{\theta_H^\gamma-\delta(\epsilon,\eta)}^{\theta_0(x)} -u(\theta,1-P(x|\theta))\, p(x|\theta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&\quad + \int_{\theta_0(x)}^{\theta_H^\gamma} -u(\theta,1-P(x|\theta))\, p(x|\theta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&\geq \int_{\theta_L^\gamma}^{\theta_H^\gamma-\delta(\epsilon,\eta)} \frac{q(\theta,x)}{q(\theta,\bar{x}+\eta)} q(\theta,\bar{x}+\eta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&\quad + \int_{\theta_H^\gamma-\delta(\epsilon,\eta)}^{\theta_0(x)} \frac{q(\theta,x)}{q(\theta,\bar{x}+\eta)} q(\theta,\bar{x}+\eta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&\quad + \frac{q(\theta_H^\gamma-\delta(\epsilon,\eta),x)}{q(\theta_H^\gamma-\delta(\epsilon,\eta),\bar{x}+\eta)} \int_{\theta_0(x)}^{\theta_H^\gamma} q(\theta,\bar{x}+\eta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&\geq \frac{q(\theta_H^\gamma-\delta(\epsilon,\eta),x)}{q(\theta_H^\gamma-\delta(\epsilon,\eta),\bar{x}+\eta)} \int_{\theta_L^\gamma}^{\theta_H^\gamma} q(\theta,\bar{x}+\eta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
&= \frac{q(\theta_H^\gamma-\delta(\epsilon,\eta),x)}{q(\theta_H^\gamma-\delta(\epsilon,\eta),\bar{x}+\eta)} \Delta S(\bar{x}+\eta) = 0.
\end{aligned}
$$

The first inequality follows from the fact that (i) for any $\theta \leq \theta_0(x)$, $u(\theta,1-P(x|\theta)) < 0$, whereas for any $\theta > \theta_0(x)$, $u(\theta,1-P(x|\theta)) > 0$, and (ii) for $\theta \in [\theta_0(x), \theta_H^\gamma]$, $\pi'(1|\theta) \leq \pi^{\epsilon,\gamma,\eta}(1|\theta)$. Together, these properties imply that

$$
\begin{aligned}
& \int_{\theta_0(x)}^{\theta_H^\gamma} -u(\theta,1-P(x|\theta))\, p(x|\theta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) \\
\geq 0 \geq \; & \frac{q(\theta_H^\gamma-\delta(\epsilon,\eta),x)}{q(\theta_H^\gamma-\delta(\epsilon,\eta),\bar{x}+\eta)} \int_{\theta_0(x)}^{\theta_H^\gamma} q(\theta,\bar{x}+\eta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta).
\end{aligned}
$$

The second inequality follows from the fact that, $\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)$ turns from positive to negative at $\theta = \theta_H^\gamma - \delta(\epsilon,\eta) \leq \theta_0(x)$, along with the fact that, for any $\theta \in [\theta_L^\gamma, \theta_0(x)]$, the function $q(\theta,x)/q(\theta,\bar{x}+\eta)$ is non-increasing in θ as implied by the log-supermodularity of $|u(\theta,1-P(x|\theta))|\, p(x|\theta)$ over $\{(\theta,x) \in [0,1] \times \mathbb{R} : u(\theta,1-P(x|\theta)) \leq 0\}$, as implied by Property (2') and the assumption that $p(x|\theta)$ is log-supermodular. Finally, the last two equalities follow from the fact that $\theta_0(\bar{x}+\eta) > \theta_0(\bar{x}) > \theta_H \geq \theta_H^\gamma$, which implies that $u(\theta,1-P(\bar{x}+\eta|\theta)) \leq 0$ for all $\theta \leq \theta_H^\gamma$, and hence that

$$
\int_{\theta_L^\gamma}^{\theta_H^\gamma} q(\theta,\bar{x}+\eta) \left(\pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta)\right) \mathrm{d}F(\theta) = \Delta S(\bar{x}+\eta)
$$

along with the fact that, by construction of the policy $\Gamma^{\epsilon,\gamma,\eta}$, $\Delta S(\bar{x}+\eta) = 0$. Hence, for any $x \in (x^*(\theta_H^\gamma - \delta(\epsilon,\eta)), x^*(\theta_H^\gamma))$, $\Delta S(x) \geq 0$, which implies that, for any x such that $(x,1)$ are mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$, $U^{\Gamma^{\epsilon,\gamma,\eta}}(x,1|x) \geq 0$.

Similar arguments imply that, for any $x \in [x^*(\theta_H^\gamma), \overline{x}+\eta]$,

$$\Delta S(x) = \int_{\theta_L^\gamma}^{\theta_H^\gamma} -u\left(\theta, 1-P\left(x|\theta\right)\right) p\left(x|\theta\right)\left(\pi'\left(1|\theta\right) - \pi^{\epsilon,\gamma,\eta}\left(1|\theta\right)\right) \mathrm{d}F\left(\theta\right)$$
$$= \int_{\theta_L^\gamma}^{\theta_H^\gamma} \tfrac{q(\theta,x)}{q(\theta,\bar{x}+\eta)} q\left(\theta, \bar{x}+\eta\right)\left(\pi'\left(1|\theta\right) - \pi^{\epsilon,\gamma,\eta}\left(1|\theta\right)\right) \mathrm{d}F\left(\theta\right) \geq \tfrac{q(\theta_H^\gamma-\delta(\epsilon,\eta),x)}{q(\theta_H^\gamma-\delta(\epsilon,\eta),\bar{x}+\eta)} \Delta S(\bar{x}+\eta) = 0,$$

implying that, for such x too, if $(x,1)$ are mutually consistent under $\Gamma^{\epsilon,\gamma,\eta}$, then $U^{\Gamma^{\epsilon,\gamma,\eta}}(x,1|x) \geq 0$. Together, the results above thus imply that, when ϵ,γ,η are small, the new policy $\Gamma^{\epsilon,\gamma,\eta} \in \mathbb{G}$.

We now show that, when property (2") in Condition M holds, the new policy yields the policy maker an expected payoff strictly higher than Γ'. To see this, observe that, fixing (γ,η), for any $\epsilon > 0$, the policy maker's payoff under the policy $\Gamma^{\epsilon,\gamma,\eta}$ is equal to

$$\begin{aligned} \mathcal{U}^P[\Gamma^{\epsilon,\gamma,\eta}] &= \int_{-\infty}^{\theta_L^\gamma+\epsilon} U^P(\theta,0)\mathrm{d}F\left(\theta\right) + \int_{\theta_H^\gamma-\delta(\epsilon,\eta)}^{\theta_H^\gamma} U^P(\theta,1)\mathrm{d}F\left(\theta\right) \\ &+ \int_{(\theta_L^\gamma+\epsilon,\theta_H^\gamma-\delta(\epsilon,\eta))\cup(\theta_H^\gamma,+\infty)} \left\{\pi'(1|\theta)U^P(\theta,1) + (1-\pi'(1|\theta))U^P(\theta,0)\right\} \mathrm{d}F\left(\theta\right). \end{aligned}$$

Differentiating $\mathcal{U}^P[\Gamma^{\epsilon,\gamma,\eta}]$ with respect to ϵ, and taking the limit as $\epsilon \to 0^+$, we have that

$$\begin{aligned} \lim_{\epsilon\to 0^+} \tfrac{d\mathcal{U}^P[\Gamma^{\epsilon,\gamma,\eta}]}{d\epsilon} &= f(\theta_H^\gamma)(1-\pi'(1|\theta_H^\gamma))\left[U^P(\theta_H^\gamma,1) - U^P(\theta_H^\gamma,0)\right] \times \lim_{\epsilon\to 0^+} \tfrac{\partial\delta(\epsilon,\eta)}{\partial\epsilon} \\ &\quad - f(\theta_L^\gamma)\pi'(1|\theta_L^\gamma)\left[U^P(\theta_L^\gamma,1) - U^P(\theta_L^\gamma,0)\right] \\ &= f(\theta_L^\gamma)\pi'(1|\theta_L^\gamma)\left(\left[U^P(\theta_H^\gamma,1) - U^P(\theta_H^\gamma,0)\right] \tfrac{p(\bar{x}+\eta|\theta_L^\gamma)u(\theta_L^\gamma,1-P(\bar{x}+\eta|\theta_L^\gamma))}{p(\bar{x}+\eta|\theta_H^\gamma)u(\theta_H^\gamma,1-P(\bar{x}+\eta|\theta_H^\gamma))} - \left[U^P(\theta_L^\gamma,1) - U^P(\theta_L^\gamma,0)\right]\right). \end{aligned}$$

Therefore, $\lim_{\epsilon\to 0^+} \frac{d\mathcal{U}^P[\Gamma^{\epsilon,\gamma,\eta}]}{d\epsilon} > 0$ if and only if

$$\frac{U^P\left(\theta_H^\gamma,1\right) - U^P\left(\theta_H^\gamma,0\right)}{U^P\left(\theta_L^\gamma,1\right) - U^P\left(\theta_L^\gamma,0\right)} > \frac{p\left(\bar{x}+\eta|\theta_H^\gamma\right) u\left(\theta_H^\gamma, 1-P\left(\bar{x}+\eta|\theta_H^\gamma\right)\right)}{p\left(\bar{x}+\eta|\theta_L^\gamma\right) u\left(\theta_L^\gamma, 1-P\left(\bar{x}+\eta|\theta_L^\gamma\right)\right)}.$$

Property (2") in Condition M, together with the fact that $\bar{x} \leq \bar{x}_G$, guarantee that the last inequality holds. We conclude that the policy $\Gamma^{\epsilon,\gamma,\eta} \in \mathbb{G}$ yields the policy maker a payoff strictly higher than Γ'. This completes the proof of Claim S1-B. $\square$

Claim S1-C. *Suppose that Condition M holds and that $\Gamma' \in \mathbb{G}$ is such that*

$$\left\{\theta \in \left(\inf \Theta\left(\bar{x}\right), \theta_H\right) : \pi'(1|\theta) > 0\right\} \text{ has zero } F\text{-measure.} \tag{S5}$$

Then, $\pi'(1|\theta) = 0$ for F-almost all $\theta \leq \theta^*$ and $\pi'(1|\theta) = 1$ for F-almost all $\theta > \theta^*$.

Proof of Claim S1-C. Condition (S5), together with the definition of θ_H and the fact that $U^{\Gamma'}\left(\bar{x},1|\bar{x}\right) = 0$, jointly imply that $\theta_H < \sup \Theta\left(\bar{x}\right)$ and that $U^{\Gamma'}\left(\bar{x},1|\bar{x}\right) = U^{\Gamma^{\theta_H}}\left(\bar{x},1|\bar{x}\right)$, where $\Gamma^{\theta_H} = \left(\{0,1\}, \pi^{\theta_H}\right)$ is the monotone deterministic policy with cut-off θ_H.[12] In other

[12] If $\theta_H \geq \sup \Theta\left(\bar{x}\right)$ then $p^{\Gamma'}\left(\bar{x},1\right) \equiv \int p(\bar{x}|\theta)\pi'(1|\theta)\mathrm{d}F(\theta) = 0$ contradicting the assumption that $U^{\Gamma'}\left(\bar{x},1|\bar{x}\right) = 0$ which requires that $(x,1)$ are mutually consistent under Γ'.

words, from the perspective of an agent with signal $\bar{x}$, the information learned under Γ', by the announcement that $s=1$ is the same as the one learnt under Γ^{θ_H}.

Suppose that $\theta_H > \theta^*$. For any deterministic monotone policy $\Gamma^{\hat{\theta}} = (\{0,1\}, \pi^{\hat{\theta}})$, any $\tilde{\theta} \geq \hat{\theta}$, let $\varphi(\tilde{\theta};\hat{\theta}) \equiv \int_{\hat{\theta}}^{\sup \Theta(x^*(\tilde{\theta}))} u(\theta, 1-P(x^*(\tilde{\theta})|\theta))p(x^*(\tilde{\theta})|\theta)\mathrm{d}F(\theta)$ and $\bar{\varphi}(\hat{\theta}) \equiv \inf_{\tilde{\theta}\geq\hat{\theta}} \varphi(\tilde{\theta};\hat{\theta})$. Note that, for any $\tilde{\theta}$ such that $(x^*(\tilde{\theta}),1)$ are mutually consistent under the policy $\Gamma^{\hat{\theta}}$, $\varphi(\tilde{\theta};\hat{\theta}) = U^{\Gamma^{\hat{\theta}}}(x^*(\tilde{\theta}),1|x^*(\tilde{\theta}))p^{\Gamma^{\hat{\theta}}}(x^*(\tilde{\theta}),1)$. We claim that, for any $\hat{\theta} > \theta^*$, $\bar{\varphi}(\hat{\theta}) > 0$. To see this, consider first the case where $\hat{\theta} \in \arg\min_{\tilde{\theta}\geq\hat{\theta}} \varphi(\tilde{\theta};\hat{\theta})$. Observe that, if each agent follows a threshold strategy with cut-off $x^*(\hat{\theta})$, then default occurs only for fundamentals weakly below $\hat{\theta}$. Because $u(\theta, 1-P(x^*(\hat{\theta})|\theta)) > 0$ for all $\theta > \hat{\theta}$ and because $p(x^*(\hat{\theta})|\theta) > 0$ in a right-neighborhood of $\hat{\theta}$, then necessarily $\bar{\varphi}(\hat{\theta}) = \varphi(\hat{\theta};\hat{\theta}) > 0$. Next, suppose that $\hat{\theta} \notin \arg\min_{\tilde{\theta}\geq\hat{\theta}} \varphi(\tilde{\theta};\hat{\theta})$. Then, observe that, for almost any $\hat{\theta} \geq \theta^*$, and any $\tilde{\theta}_m \in \arg\min_{\tilde{\theta}\geq\hat{\theta}} \varphi(\tilde{\theta};\hat{\theta})$, with $\tilde{\theta}_m > \hat{\theta}$,[13] $\partial\varphi(\tilde{\theta}_m;\hat{\theta})/\partial\hat{\theta} = -u(\hat{\theta}, 1-P(x^*(\tilde{\theta}_m)|\hat{\theta}))p(x^*(\tilde{\theta}_m)|\hat{\theta})f(\hat{\theta}) \geq 0$, where the inequality follows from the fact that $u(\hat{\theta}, 1-P(x^*(\tilde{\theta}_m)|\hat{\theta})) < 0$ which, in turn, is a consequence of (i) the definition of $x^*(\cdot)$ and (ii) the fact that $\tilde{\theta}_m > \hat{\theta}$.

By the definition of θ^*, $\bar{\varphi}(\theta^*) = 0$, and $d\bar{\varphi}(\theta^*)/d\hat{\theta} > 0$. The above properties thus imply that, for any $\hat{\theta} > \theta^*$, $\bar{\varphi}(\hat{\theta}) > 0$, as claimed.

By the definition of $\bar{x}$, $U^{\Gamma'}(\bar{x},1|\bar{x}) = 0$. Under Condition (S5), this implies that, when agents pledge for $x > \bar{x}$ and refrain from pledging for $x < \bar{x}$, the default outcome $\theta_0(\bar{x})$ must necessarily satisfy $\theta_0(\bar{x}) > \theta_H$, for, otherwise, an agent with signal $\bar{x}$ would strictly prefer pledging to not pledging. Because $U^{\Gamma'}(\bar{x},1|\bar{x}) = U^{\Gamma^{\theta_H}}(\bar{x},1|\bar{x})$, that $\theta_0(\bar{x}) > \theta_H > \theta^*$, along with the fact that $\varphi(\theta_0(\bar{x});\theta_H) > 0$, however, implies that $U^{\Gamma'}(\bar{x},1|\bar{x}) > 0$, a contradiction.

Hence, it must be that $\theta_H \leq \theta^*$. However, by definition of θ^*, if $\theta_H < \theta^*$, then there exists $\theta > \theta_H$ such that $(x^*(\theta),1)$ are mutually consistent under Γ^{θ_H} and such that

$$U^{\Gamma^{\theta_H}}(x^*(\theta),1|x^*(\theta))p^{\Gamma^{\theta_H}}(x^*(\theta),1) = \int_{\theta_H}^{\sup \Theta(x^*(\theta))} u(\tilde{\theta}, 1-P(x^*(\theta)|\tilde{\theta}))p(x^*(\theta)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}) < 0.$$

Now note that

$$\begin{aligned} U^{\Gamma'}(x^*(\theta),1|x^*(\theta))\,p^{\Gamma'}(x^*(\theta),1) &= \int_{\inf \Theta(x^*(\theta))}^{\theta_H} u(\tilde{\theta}, 1-P(x^*(\theta)|\tilde{\theta}))\pi'(1|\tilde{\theta})p(x^*(\theta)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}) \\ &\quad + \int_{\theta_H}^{\sup \Theta(x^*(\theta))} u(\tilde{\theta}, 1-P(x^*(\theta)|\tilde{\theta}))p(x^*(\theta)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}) \end{aligned}$$

with $p^{\Gamma'}(x^*(\theta),1) = \int_{\inf \Theta(x^*(\theta))}^{\theta_H} \pi'(1|\tilde{\theta})p(x^*(\theta)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}) + p^{\Gamma^{\theta_H}}(x^*(\theta),1) > 0$. Because, for any $\tilde{\theta} < \theta_H$, $u(\tilde{\theta}, 1-P(x^*(\theta)|\tilde{\theta})) < 0$, we thus have that $U^{\Gamma'}(x^*(\theta),1|x^*(\theta)) < 0$. But this contradict the assumption that $\Gamma' \in \mathbb{G}$. We thus conclude that necessarily $\theta_H = \theta^*$. Furthermore, because $\{\theta \in (\inf \Theta(\bar{x}), \theta_H) : \pi'(1|\theta) > 0\}$ has 0 F-measure, it must be that

[13] Note that $\varphi(\tilde{\theta};\hat{\theta})$ is absolutely continuous in $\hat{\theta}$, and therefore is differentiable in $\hat{\theta}$ almost everywhere.

$U^{\Gamma'}(\bar{x},1|\bar{x}) = U^{\Gamma^{\theta^*}}(\bar{x},1|\bar{x})$. Furthermore, because $\theta_0(\bar{x}) > \theta^*$, we also have that $U^{\Gamma^0}(\bar{x},1|\bar{x}) \leq U^{\Gamma^{\theta^*}}(\bar{x},1|\bar{x}) = 0$. Hence, $\bar{x} \leq \bar{x}_G$, which, by virtue of Property 1 in Condition M, implies that $\inf \Theta(\bar{x}) \leq 0$. Condition (S5), along with the fact that $\pi'(1|\theta) = 0$ for all $\theta \leq 0$ and $\pi'(1|\theta) = 1$ for F-almost all $\theta > \theta_H = \theta^*$, thus imply that Γ' is such that $\pi'(1|\theta) = 0$ for F-almost all $\theta \leq \theta^*$ and $\pi'(1|\theta) = 1$ for F-almost all $\theta > \theta^*$. This completes the proof of Claim S1-C. □

Step 2. Step 1 implies that $\arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\} \neq \emptyset$ and that any $\Gamma \in \arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\}$ is such that $\pi(1|\theta) = 0$ for F-almost all $\theta \leq \theta^*$ and $\pi(1|\theta) = 1$ for F-almost all $\theta > \theta^*$. The result in the theorem then follows from observing that, given any $\Gamma \in \arg\max_{\tilde{\Gamma}\in\mathbb{G}}\{\mathcal{U}^P[\tilde{\Gamma}]\}$, there exists a nearby deterministic monotone policy $\Gamma^{\hat{\theta}} \in \mathbb{G}$ with cut-off $\hat{\theta} = \theta^* + \tilde{\varepsilon}$, for $\tilde{\varepsilon} > 0$ but small, such that $\Gamma^{\hat{\theta}}$ satisfies PCP (i.e., $U^{\Gamma^{\hat{\theta}}}(x,1|x) > 0$ all x such that $(x,1)$ are mutually consistent under $\Gamma^{\hat{\theta}}$) and yields the policy maker a payoff arbitrarily close to that under Γ. Q.E.D.

Section S2: Proof of Example 2 in the main text

The proof is in two steps. Step 1 characterizes the threshold θ^*_σ defining the optimal deterministic monotone rule, whereas Step 2 constructs the non-monotone policy that strictly improves over the optimal deterministic monotone one.

Step 1. The primitives in this example satisfy the conditions in Theorem 2 in the main text. This means that, given any signal s disclosed by any policy Γ, MARP is in threshold strategies, which in turn implies that the default outcome is monotone in θ.

Next recall that, for any default threshold $\theta \in [0,1]$, the corresponding signal threshold $x^*_\sigma(\theta)$ is implicitly defined by $P_\sigma(x^*_\sigma(\theta)|\theta) = \theta$. Using the fact that, for any $\theta \in [-K, 1+K]$ and $x \in [\theta-\sigma, \theta+\sigma]$, $P_\sigma(x|\theta) = (x-\theta+\sigma)/2\sigma$, we have that $x^*_\sigma(\theta) = (1+2\sigma)\theta - \sigma$.

For any $\hat{\theta} \in [0,1]$, let $\Gamma^{\hat{\theta}} \equiv \{\{0,1\}, \pi^{\hat{\theta}}\}$ be the deterministic monotone policy with cutoff $\hat{\theta}$. Next, for any $\theta \in [\hat{\theta}/(1+2\sigma), 1]$, let $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) \equiv U^{\Gamma^{\hat{\theta}}}_\sigma(x^*_\sigma(\theta), 1|x^*_\sigma(\theta))$ be the expected payoff differential between pledging and not pledging of the marginal agent with signal $x^*_\sigma(\theta)$, when each agent pledges if and only if their signal is above $x^*_\sigma(\theta)$ (and hence default occurs if, and only if, fundamentals are below θ), the quality of the agents' signal is σ, and the policy $\Gamma^{\hat{\theta}}$ announces that $s = 1$, thus revealing that $\theta \geq \hat{\theta}$. Note that, for any $0 \leq \theta < \hat{\theta}/(1+2\sigma)$, $x^*_\sigma(\theta) + \sigma < \hat{\theta}$, which implies that the signal $x^*_\sigma(\theta)$ is not consistent with the event that fundamentals are above $\hat{\theta}$. Equivalently, when $\theta \geq \hat{\theta}$, the lowest possible signal that an individual may receive is $\hat{\theta} - \sigma$. When each agent pledges if and only if $x > \hat{\theta} - \sigma$, default occurs if and only if $\theta \leq \hat{\theta}/(1+2\sigma)$. Hence, the lowest default threshold that is consistent with the policy $\Gamma^{\hat{\theta}}$ is $\hat{\theta}/(1+2\sigma)$. The function $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta)$ is thus defined only for $\theta \in [\hat{\theta}/(1+2\sigma), 1]$.

The cutoff θ^*_σ characterizing the optimal deterministic monotone policy is given by

$$\theta^*_\sigma = \inf\{\hat{\theta} \in [0,1] : V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) \geq 0 \;\text{ for all } \theta \in [\hat{\theta}/(1+2\sigma), 1]\}. \tag{S6}$$

Claim S2-1. *For any $\hat{\theta} \in [0,1]$, $V_\sigma^{\Gamma^{\hat{\theta}}}(\cdot)$ has a unique minimizer. Letting $\theta_\sigma^{\min}(\hat{\theta}) \equiv$ arg $min_{\theta\in[\hat{\theta}/(1+2\sigma),1]} V_\sigma^{\Gamma^{\hat{\theta}}}(\theta)$, we have that $\theta_\sigma^{\min}(\hat{\theta})$ satisfies $x^*_\sigma(\theta_\sigma^{\min}(\hat{\theta})) - \sigma = \hat{\theta}$.*

Proof of Claim S2-1. Clearly, for any $\theta \in [\hat{\theta}/(1+2\sigma), \hat{\theta}]$, $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g$. This is because when each agent pledges if and only if $x > x^*_\sigma(\theta)$ default occurs only for fundamentals below θ. Hence the announcement that $\theta > \hat{\theta}$ reveals to the marginal agent with signal $x^*_\sigma(\theta)$ that default will not occur.

Next, observe that for any $\theta \in (\hat{\theta}, (\hat{\theta}+2\sigma)/(1+2\sigma)]$, $x^*_\sigma(\theta) - \sigma < \hat{\theta}$, implying that[14]

$$V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma\{\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x^*_\sigma(\theta)\} = g - (g+|b|)\,\frac{\theta - \hat{\theta}}{(1+2\sigma)\,\theta - \hat{\theta}},$$

which is strictly decreasing in θ. Finally, note that, for any $\theta \in ((\hat{\theta}+2\sigma)/(1+2\sigma), 1]$, $x^*_\sigma(\theta) - \sigma > \hat{\theta}$, implying that

$$V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma\{\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x^*_\sigma(\theta)\} = g + (g+|b|)(\theta - 1),$$

which is strictly increasing in θ. Hence, $V_\sigma^{\Gamma^{\hat{\theta}}}(\cdot)$ has a single minimizer over $[\hat{\theta}/(1+2\sigma), 1]$. The latter is equal to $\theta_\sigma^{min}(\hat{\theta}) = (\hat{\theta}+2\sigma)/(1+2\sigma)$ and is such that $x^*_\sigma(\theta_\sigma^{min}(\hat{\theta})) - \sigma = \hat{\theta}$. $\square$

Next, let $\Gamma^{\theta^*_\sigma} \equiv (\{0,1\}, \pi^{\theta^*_\sigma})$ be the optimal deterministic monotone policy (with cut-off $\hat{\theta} = \theta^*_\sigma$). Using the characterization of θ^*_σ in (S6), we thus have that, under $\Gamma^{\theta^*_\sigma}$, at the point $\theta_\sigma^{min}(\theta^*_\sigma)$ at which $V_\sigma^{\Gamma^{\theta^*_\sigma}}$ reaches its minimum, $V_\sigma^{\Gamma^{\theta^*_\sigma}}(\theta_\sigma^{min}(\theta^*_\sigma)) = 0$. Using the fact that

$$V_\sigma^{\Gamma^{\theta^*_\sigma}}\left(\theta_\sigma^{min}(\theta^*_\sigma)\right) = g - (g+|b|)\,\frac{\theta_\sigma^{min}(\theta^*_\sigma) - \theta^*_\sigma}{(1+2\sigma)\,\theta_\sigma^{min}(\theta^*_\sigma) - \theta^*_\sigma},$$

we then have that $\theta^*_\sigma = (1+2\sigma)\frac{|b|}{g+|b|} - 2\sigma$. Next, let $\Gamma_\emptyset$ be the no-disclosure policy and note that, for any $\theta \in [0,1]$,

$$V_\sigma^{\Gamma_\emptyset}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma\{\tilde{\theta} \leq \theta | x^*_\sigma(\theta)\} = g + (g+|b|)(\theta - 1),$$

which is increasing in θ and has a unique zero at $\theta = |b|/(g+|b|) \equiv \theta^{MS}$.

This means that, in the absence of any disclosure, under the unique rationalizable strategy profile (and hence under MARP), each agent pledges if and only if $x > x^*_\sigma(\theta^{MS})$, and default

[14] The notation $\mathbb{P}_\sigma\left\{\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x\right\}$ stands for the probability that an agent with signal x assigns to the event that $\tilde{\theta} \leq \theta$ when the quality of his exogenous signal is parametrized by σ and the policy reveals that $\tilde{\theta} \geq \hat{\theta}$.

occurs if and only if fundamentals are below θ^{MS}. The results above then imply that the optimal deterministic policy $\Gamma^{\theta_\sigma^*}$ is defined by a threshold $\theta_\sigma^* = (1+2\sigma)\,\theta^{MS} - 2\sigma = x_\sigma^*\left(\theta^{MS}\right) - \sigma$ that coincides with the left end-point of the support of the posterior beliefs of each agent with signal $x_\sigma^*\left(\theta^{MS}\right)$. In fact, for any truncation point $\hat{\theta} < x_\sigma^*\left(\theta^{MS}\right) - \sigma$, there exists θ close to θ^{MS} such that $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) < 0$ implying that refraining from pledging for all $x < x_\sigma^*\left(\theta^{MS}\right)$ is rationalizable in the continuation game following the announcement that $\theta \geq \hat{\theta}$, implying that the policy $\Gamma^{\hat{\theta}}$ fails to satisfy PCP. Similarly, for any truncation point $\hat{\theta} > x_\sigma^*(\theta^{MS}) - \sigma$, $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta)$ reaches its minimum at $\theta_\sigma^{min}(\hat{\theta}) > \theta^{MS}$ and is such that $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta_\sigma^{min}(\hat{\theta})) = V_\sigma^{\Gamma_\emptyset}(\theta_\sigma^{min}(\hat{\theta})) > V_\sigma^{\Gamma_\emptyset}(\theta^{MS}) = 0$, where the inequality follows from the monotonicity of $V_\sigma^{\Gamma_\emptyset}(\cdot)$. Hence, $\theta_\sigma^* = x_\sigma^*\left(\theta^{MS}\right) - \sigma$.

Step 2. Having characterized the optimal deterministic monotone policy $\Gamma^{\theta_\sigma^*}$, we now show that, when σ is small, there exists another policy Γ that also satisfies PCP and guarantees no default for a larger set of fundamentals than $\Gamma^{\theta_\sigma^*}$.

Let $\sigma^{\#} \equiv \frac{\theta^{MS}}{2(1-\theta^{MS})} > 0$. For any $\sigma \in (0, \sigma^{\#})$, $\theta_\sigma^* = (1+2\sigma)\,\theta^{MS} - 2\sigma > 0$. For any $\sigma, \delta, \gamma > 0$ small, let $\theta_\sigma''(\delta,\gamma) \equiv x_\sigma^*\left(\theta^{MS} - \delta\right) - \sigma = (1+2\sigma)(\theta^{MS} - \delta) - 2\sigma$ and $\theta_\sigma'(\delta,\gamma) \equiv \theta_\sigma''(\delta,\gamma) - \gamma$. Note that, for any $\sigma \in (0,\sigma^{\#})$, $\delta > 0$ and $\gamma > 0$ can be chosen so that $0 < \theta'_\sigma(\delta,\gamma) < \theta_\sigma''(\delta,\gamma) < \theta_\sigma^*$.

Consider the non-monotone deterministic policy $\Gamma_{\delta,\gamma} \equiv \{\{0,1\}, \pi_{\delta,\gamma}\}$ given by

$$\pi_{\delta,\gamma}(1|\theta) \equiv \mathbf{1}\left\{\theta \in \left[\theta'_\sigma(\delta,\gamma), \theta_\sigma''(\delta,\gamma)\right] \cup [\theta_\sigma^*, \infty)\right\}.$$

We show that, for any $\sigma \in (0, \sigma^{\#})$, there exit $\delta, \gamma > 0$ such that (i) $0 \leq \theta'_\sigma(\delta,\gamma) < \theta_\sigma''(\delta,\gamma) < \theta_\sigma^*$, and (ii) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) \geq 0$ for all $\theta > \theta'_\sigma(\delta,\gamma)/(1+2\sigma)$, with $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = 0$ only for $\theta = \theta^{MS}$.[15]

First observe that, for any $\sigma \in (0, \sigma^{\#})$, $\delta \in \left(0, \theta^{MS} - \frac{2\sigma}{1+2\sigma}\right)$ and

$$0 < \gamma \leq (1+2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma \equiv R_0\left(\delta, \theta^{MS}, \sigma\right)$$

guarantee that $0 \leq \theta'_\sigma(\delta,\gamma) < \theta_\sigma''(\delta,\gamma) < \theta_\sigma^*$.[16]

Next note that, for any (σ, δ, γ) with $\sigma \in (0, \sigma^{\#})$, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1+2\sigma)\right)$ and $0 < \gamma \leq R_0\left(\delta, \theta^{MS}, \sigma\right)$, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = V_\sigma^{\Gamma^{\theta_\sigma^*}}(\theta)$ for all $\theta \in \left[\theta^{MS} - \delta, 1\right]$. Indeed, for any $\theta \in \left[\theta^{MS} - \delta, 1\right]$, $x_\sigma^*(\theta) - \sigma > \theta_\sigma''(\delta,\gamma)$ implying that the the posterior beliefs of the marginal agent with signal $x_\sigma^*(\theta)$ under the policy $\Gamma_{\delta,\gamma}$ coincide with those under the policy $\Gamma^{\theta_\sigma^*}$.

[15] Consistently with the notation above, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)$ is the expected payoff of the marginal agent with signal $x_\sigma^*(\theta)$ when the policy $\Gamma_{\delta,\gamma}$ announces that $s = 1$ and the quality of the agents' exogenous signals is parametrized by σ. For any $\theta < \theta'_\sigma(\delta,\gamma)/(1+2\sigma)$, $x_\sigma^*(\theta) + \sigma < \theta'$, which implies that the signal $x_\sigma^*(\theta)$ is not consistent with the event that fundamentals are above $\theta'_\sigma(\delta,\gamma)$. Equivalently, because the lowest signal that is consistent with $\theta \in [\theta'_\sigma(\delta,\gamma), \theta_\sigma''(\delta,\gamma)] \cup [\theta_\sigma^*, \infty)$ is $\theta'_\sigma(\delta,\gamma) - \sigma$, the lowest default threshold is $\theta'_\sigma(\delta,\gamma)/(1+2\sigma)$.

[16] Observe that $\sigma \in (0,\sigma^{\#})$ implies that $\theta^{MS} - 2\sigma/(1+2\sigma) > 0$. In turn, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1+2\sigma)\right)$ implies that $0 < \theta_\sigma''(\delta,\gamma) < \theta_\sigma^*$ and that $R_0\left(\delta, \theta^{MS}, \sigma\right) > 0$. Finally, that $0 < \gamma \leq R_0\left(\delta, \theta^{MS}, \sigma\right)$ implies that $0 \leq \theta'_\sigma(\delta,\gamma) < \theta_\sigma''(\delta,\gamma)$.

Let $\theta^{\sharp}_{\sigma}(\delta,\gamma)$ be such that $x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}(\delta,\gamma)\right)-\sigma=\theta^{'}_{\sigma}(\delta,\gamma)$. Dropping the arguments of $\theta^{\sharp}_{\sigma}(\delta,\gamma)$, $\theta^{'}_{\sigma}(\delta,\gamma)$ and $\theta^{''}_{\sigma}(\delta,\gamma)$ to ease the notation, we have that

$$\theta'=\theta''-\gamma=x^*_{\sigma}\left(\theta^{MS}-\delta\right)-\sigma-\gamma=(1+2\sigma)\left(\theta^{MS}-\delta\right)-2\sigma-\gamma.$$

From the definition of $\hat{\theta}$ we have that $x^*_{\sigma}(\hat{\theta})-\sigma=(1+2\sigma)\,\theta^{\sharp}-2\sigma=\theta'$. Combining the above two results we obtain that $\theta^{\sharp}=\theta^{MS}-\delta-\gamma/(1+2\sigma)$. Fixing $\sigma\in(0,\sigma^{\#})$, note that, for $\delta,\gamma>0$ small, $\theta^{\sharp}\geq\theta^*_{\sigma}$. Specifically, for any $\sigma\in(0,\sigma^{\#})$ and any $0<\delta<2\sigma\left(1-\theta^{MS}\right)$, $\theta^{\sharp}\geq\theta^*_{\sigma}$ if and only if

$$\gamma\leq(1+2\sigma)\left(2\sigma\left(1-\theta^{MS}\right)-\delta\right)\equiv R_1\left(\delta,\theta^{MS},\sigma\right).$$

Next, observe that, for any $\theta\in\left[\theta^{\sharp},\theta^{MS}-\delta\right)$,

$$\begin{aligned}V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[x^*_{\sigma}(\theta)-\sigma,\theta'']\cup[\theta^*_{\sigma},\infty);x^*_{\sigma}(\theta)\}\\ &= g-(g+|b|)\left(\theta''-\theta^*_{\sigma}+2\sigma(1-\theta)\right)/\left(\theta''-\theta^*_{\sigma}+2\sigma\right),\end{aligned}$$

which is strictly increasing in θ. Similarly, for any $\theta\in\left[\theta^*_{\sigma},\theta^{\sharp}\right)$,

$$\begin{aligned}V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^*_{\sigma},\infty);x^*_{\sigma}(\theta)\}\\ &= g-(g+|b|)\,\frac{\theta-\theta^*_{\sigma}+\gamma}{x^*_{\sigma}(\theta)+\sigma-\theta^*_{\sigma}+\gamma}=g-(g+|b|)\,\frac{\theta-\theta^*_{\sigma}+\gamma}{(1+2\sigma)\,\theta-\theta^*_{\sigma}+\gamma},\end{aligned}$$

which is strictly deceasing for any $\gamma\leq\theta^*_{\sigma}$. Note that $\theta'\geq0$ requires that $\gamma\leq\theta^*_{\sigma}$. Next, note that, for $\theta\in[\theta'',\theta^*_{\sigma})$,

$$\begin{aligned}V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^*_{\sigma},\infty);x^*_{\sigma}(\theta)\}\\ &= g-(g+|b|)\,\frac{\gamma}{x^*_{\sigma}(\theta)+\sigma-\theta^*_{\sigma}+\gamma}=g-(g+|b|)\,\frac{\gamma}{(1+2\sigma)\,\theta-\theta^*_{\sigma}+\gamma},\end{aligned}$$

and, therefore, $V^{\Gamma_{\delta,\gamma}}_{\sigma}(\cdot)$ is increasing over the range $[\theta'',\theta^*_{\sigma})$. Finally, for $\theta\in[\theta',\theta'')$, we have that

$$\begin{aligned}V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^*_{\sigma},\infty);x^*_{\sigma}(\theta)\}\\ &= g-(g+|b|)\,\frac{\theta-\theta'}{x^*_{\sigma}(\theta)+\sigma-\theta^*_{\sigma}+\gamma}=g-(g+|b|)\,\frac{\theta-\theta'}{(1+2\sigma)\,\theta-\theta^*_{\sigma}+\gamma}.\end{aligned}$$

Hence $V^{\Gamma_{\delta,\gamma}}_{\sigma}(\cdot)$ is decreasing over $[\theta',\theta'')$ if $(1+2\sigma)\,\theta'=x^*_{\sigma}(\theta')+\sigma>\theta^*_{\sigma}$. Using the fact that $\theta'=\theta''-\gamma$, together with the fact that $\theta''=x^*_{\sigma}\left(\theta^{MS}-\delta\right)-\sigma$ and $\theta^*_{\sigma}=(1+2\sigma)\,\theta^{MS}-2\sigma$, we have that $(1+2\sigma)\,\theta'>\theta^*_{\sigma}$ if

$$\gamma\ <\ 2\sigma\left[(1+2\sigma)\,\theta^{MS}-2\sigma\right]/(1+2\sigma)-(1+2\sigma)\,\delta\equiv R_2\left(\delta,\theta^{MS},\sigma\right).$$

Lastly, observe that, for any $\theta\in[\theta^{'}/(1+2\sigma),\theta']$, $V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta)=g$.

We thus have that the function $V_\sigma^{\Gamma_{\delta,\gamma}}$ is such that (1) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) \geq 0$ for all $\theta \geq \theta'/(1+2\sigma)$, and (2) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = 0$ only if $\theta = \theta^{MS}$, if and only if the following conditions hold: (a) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta^\sharp) > 0$, and (b) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'') > 0$. Requiring that $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta^\sharp) > 0$ is equivalent to

$$g - (g + |b|)\left(\theta^\sharp - \theta^*_\sigma + \gamma\right) / \left(x^*_\sigma\left(\theta^\sharp\right) + \sigma - \theta^*_\sigma + \gamma\right) \quad > \quad 0 \Leftrightarrow \theta^{MS}\left(\theta^*_\sigma - \gamma\right) - \left((1+2\sigma)\,\theta^{MS} - 2\sigma\right)\theta^\sharp > 0.$$

Recall that $\theta^*_\sigma = (1+2\sigma)\,\theta^{MS} - 2\sigma$. Using the fact that $\theta^\sharp = \theta^{MS} - \delta - \frac{\gamma}{1+2\sigma}$, we conclude that a sufficient condition for $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta^\sharp) > 0$ is that

$$\begin{aligned}\left(\theta^{MS} - \delta - \gamma/(1+2\sigma)\right)\theta^*_\sigma \quad &< \quad \theta^{MS}\left(\theta^*_\sigma - \gamma\right) \\ \Leftrightarrow \gamma \quad &< \quad \delta\,(1+2\sigma)\left((1+2\sigma)\,\theta^{MS} - 2\sigma\right)/(2\sigma) \equiv R_3\left(\delta, \theta^{MS}, \sigma\right).\end{aligned}$$

Next, observe that $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'') > 0$ is equivalent to

$$\begin{aligned}\gamma \quad &< \quad \left(1 - \theta^{MS}\right)\left((1+2\sigma)\,\theta'' - \theta^*_\sigma + \gamma\right) \\ \Leftrightarrow \gamma \quad &< \quad \left(\frac{1-\theta^{MS}}{\theta^{MS}}\right)\left((1+2\sigma)\left[(1+2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma\right] - (1+2\sigma)\,\theta^{MS} + 2\sigma\right) \equiv R_4\left(\delta, \theta^{MS}, \sigma\right).\end{aligned}$$

We conclude that, for any $\sigma \in (0, \sigma^{\#})$, (i) $0 \leq \theta'_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$, and (ii) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) \geq 0$ for all $\theta > \theta'_\sigma(\delta,\gamma)/(1+2\sigma)$, with $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = 0$ only for $\theta = \theta^{MS}$, if

$$0 < \delta < \min\left\{\theta^{MS} - \frac{2\sigma}{1+2\sigma}, 2\sigma\left(1 - \theta^{MS}\right), \frac{2\sigma\left[(1+2\sigma)\,\theta^{MS} - 2\sigma\right]}{(1+2\sigma)^2}, \frac{2\sigma}{1+2\sigma}\left[\theta^{MS} - \frac{2\sigma}{1+2\sigma}\right]\right\} \equiv \varsigma\left(\theta^{MS}, \sigma\right)$$

and $0 < \gamma < \min_{0\leq i\leq 4} R_i\left(\delta, \theta^{MS}, \sigma\right)$. Note that $\sigma < \sigma^{\#}$ implies that $\varsigma\left(\theta^{MS}, \sigma\right) > 0$, whereas $\delta < \varsigma\left(\theta^{MS}, \sigma\right)$ implies that $\min_{0\leq i\leq 4} R_i\left(\delta, \theta^{MS}, \sigma\right) > 0$. Finally note that, for any $\sigma \in (0, \sigma^{\#})$, and any $\theta \geq \theta'_\sigma(\delta,\gamma)$, the payoff $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)$ is continuous in the threshold θ^*_σ. Hence there exists a policy Γ whose rule π is given by $\pi(1|\theta) \equiv \mathbf{1}\{\theta \in [\theta'_\sigma(\delta,\gamma), \theta''_\sigma(\delta,\gamma)] \cup [\theta^*_\sigma + \varepsilon, \infty)\}$ with $\varepsilon > 0$ arbitrarily small, such that Γ strictly improves over $\Gamma^{\theta^*_\sigma}$ and is such that $V_\sigma^{\Gamma}(\theta) > 0$ for all $\theta > \theta'_\sigma(\delta,\gamma)/(1+2\sigma)$, implying that Γ satisfies PCP. Q.E.D.

Section S3: Proof of Example 3 in Main Text

Preliminaries. For any $\theta \in (0,1)$, any $\sigma \in \mathbb{R}_+$, note that, in this example $x^*_\sigma(\theta) \equiv \theta + \sigma\Phi^{-1}(\theta)$, where Φ is the cdf of the standard Normal distribution and ϕ its density. Also let $x^*_\sigma(0) \equiv -\infty$ and $x^*_\sigma(1) \equiv +\infty$. For any $(\theta_0, \hat{\theta}, \sigma) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$, let $\psi(\theta_0, \hat{\theta}, \sigma)$ denote the payoff from pledging of an investor with private signal $x^*_\sigma(\theta_0)$, when default occurs if and only if $\theta \leq \theta_0$, the policy reveals that $\theta \geq \hat{\theta}$, and the precision of private information is σ^{-2}. Then let $\hat{\sigma} \equiv \inf\{\sigma \in \mathbb{R}_+ :\ \psi(\theta_0, 0, \sigma) > 0 \text{ all } \theta_0 \in (0,1)\}$ if $\{\sigma \in \mathbb{R}_+ :\ \psi(\theta_0, 0, \sigma) > 0 \text{ all } \theta_0 \in (0,1)\} \neq \emptyset$

and else $\hat{\sigma} = +\infty$.[17] Then let $\Psi(\sigma) \equiv \inf_{\theta_0 \in (0,1)} \psi(\theta_0, 0, \sigma)$ and note that $\lim_{\sigma \to 0^+} \Psi(\sigma) < 0$, implying that $\hat{\sigma} > 0$. For any $\sigma \in \mathbb{R}_+$ for which $\psi(\theta_0, 0, \sigma) > 0$ for all $\theta_0 \in (0,1)$, the policy maker can avoid default for every $\theta > 0$ by using the monotone rule $\pi(\theta) = \mathbf{1}\{\theta > 0\}$. This case is uninteresting. Hereafter, we thus confine attention to the case in which $\sigma < \hat{\sigma}$.

Let $U_\sigma^\Gamma(x, 1|x)$ denote the payoff from pledging of an agent with signal x who expects all other agents to pledge if and only if their signal exceeds x , when the precision of private information is σ^{-2}, and the policy Γ announces that $s = 1$. Also let $U_\sigma^\Gamma(x_\sigma^*(0), 1|x_\sigma^*(0)) \equiv \lim_{x \to -\infty} U_\sigma^\Gamma(x, 1|x)$ and $U_\sigma^\Gamma(x_\sigma^*(1), 1|x_\sigma^*(1)) \equiv \lim_{x \to +\infty} U_\sigma^\Gamma(x, 1|x)$.

Now let $\mathbb{G}_\sigma$ denote the set of deterministic binary policies $\Gamma = (\{0,1\}, \pi)$ such that, $\pi(\theta) = 0$ for all $\theta \leq 0$, $\pi(\theta) = 1$ for all $\theta > 1$ and $U_\sigma^\Gamma(x, 1|x) \geq 0$ for all $x \in \mathbb{R}$.[18] From the proofs of Theorems 1 and 2, observe that, given any σ, any deterministic binary policy Γ satisfying PCP and such that $\pi(\theta) = 0$ for all $\theta \leq 0$ and $\pi(\theta) = 1$ for all $\theta > 1$ belongs in $\mathbb{G}_\sigma$. However, $\mathbb{G}_\sigma$ contains also policies that do not satisfy PCP.[19]

Proof Structure. The proof is in four steps. Step 1 establishes that, when σ is small, under any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, any interval $(\theta', \theta''] \subset \left(0, \theta^{MS}\right]$ receiving a pass grade (i.e., such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$) has a sufficiently small Lebesgue measure, with the measure vanishing as $\sigma \to 0^+$.

Step 2 then considers an *auxiliary game* G_σ in which the agents play less aggressively than under MARP. Namely, G_σ is the game in which (i) the policy maker's choice set is $\mathbb{G}_\sigma$ and (ii) given *any* policy $\Gamma \in \mathbb{G}_\sigma$, all agents pledge after receiving the signal $s = 1$ and refrain from pledging after receiving the signal $s = 0$.[20] We show that, when σ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (\underline{\theta}, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^\# \in \mathbb{G}_\sigma$ that gives a pass grade to a F-positive measure subset of $(\theta', \theta'']$, has a mesh smaller than Γ, and is such that, when agents play as in G_σ, the probability of default under $\Gamma^\#$ is strictly smaller than under Γ.

Step 3 then combines the results from Steps 1 and 2 to show that, when σ is small, given

[17]Recall that, when the announcement that $s = 1$ reveals that $\theta \geq 0$, the unique rationalizable profile features all agents pledging, irrespective of x, if and only if $\psi(\theta_0, 0, \sigma) > 0$ for all $\theta_0 \in (0,1)$. This follows directly from Lemma 1 in the main text.

[18]Consistently with the notation in the main text, we let $\pi(\theta) = 1$ (alternatively, $\pi(\theta) = 0$) denote the degenerate lottery assigning measure 1 to $s = 1$ (alternatively, $s = 0$).

[19]These are those for which there exists x such that $U_\sigma^\Gamma(x, 1|x) = 0$; in the continuation game that starts after Γ announces $s = 1$, in addition to the rationalizable profile under which all agents pledge, there also exists a rationalizable profile under which each agent pledges if and only if his signal exceeds x.

[20]The agent's behavior is consistent with MARP only for those $\Gamma \in \mathbb{G}_\sigma$ for which, for all x, $U_\sigma^\Gamma(x, 1|x) > 0$. For those $\Gamma \in \mathbb{G}_\sigma$ for which, instead, there exists x such that $U_\sigma^\Gamma(x, 1|x) = 0$, the agents' behavior is less aggressive than under MARP.

any policy $\Gamma \in \mathbb{G}_\sigma$ for which the mesh $M(\Gamma)$ of $(0, \theta^{MS}]$ is larger than ε, there exists another policy $\Gamma' \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma')$ smaller than ε such that, when agents play as in G_σ, the probability of default is strictly smaller under Γ' than under Γ. Starting from $\Gamma' \in \mathbb{G}_\sigma$ one can then construct a "nearby" policy $\Gamma^* \in \mathbb{G}_\sigma$ such that the probability of default under Γ^* is arbitrarily close to that under Γ' (and hence strictly smaller than under the original policy Γ) and such that $U_\sigma^{\Gamma^*}(x, 1|x) > 0$ for all x. The last property implies that Γ^* satisfies PCP also when agents play according to MARP. The policy Γ^* thus strictly improves upon Γ also in the original game, as claimed in the main text.

Finally, step 4 closes the proof by showing how to construct the function $\mathcal{E}$ relating the noise σ in the agents' exogenous private information to the bound $\mathcal{E}(\sigma)$ on the mesh of the policies.

Step 1. We start with the following result:

Lemma S3-A. *For any $\varepsilon \in \mathbb{R}_{++}$, there exists $\sigma(\varepsilon) \in \mathbb{R}_{++}$ such that, for any $\sigma \in (0, \sigma(\varepsilon)]$, the following is true: for any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $\pi(\theta) = 0$.*[21]

Proof of Lemma S3-A. We first show (Property S3-A below) that, for any $\sigma > 0$, if the policy maker were to replace Γ with the cutoff policy $\Gamma^{\theta'}$, then for any $\theta \leq \theta''$, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.[22] Next, we show (Property S3-B below) that, for any $\theta > \theta'$, as σ goes to zero, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ converges uniformly to $\int_0^1 u(\theta, A)dA$. Because $\int_0^1 u(\theta, A)dA < 0$ for $\theta < \theta^{MS}$, the above two properties imply that, for σ small, $U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta) < 0$ for some $\theta \in (\theta', \theta'']$, and hence that $\Gamma \notin \mathbb{G}_\sigma$. The result in the lemma then follows by contrapositive.

Property S3-A. *For any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$, $U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \leq U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ for all $\theta \leq \theta''$.*

Proof of Property S3-A. The proof follows from Results S3-A-1 and S3-A-2 below.

Result S3-A-1. *Pick any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \{d_i = (\underline{\theta}_i, \bar{\theta}_i] : i = 1, ..., N\}$ of $(0, \theta^{MS}]$ induced by Γ, take any cell $d_i = (\underline{\theta}_i, \bar{\theta}_i]$ for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_L^i = \{\{0,1\}, \pi_L^i\} \in \mathbb{G}_\sigma$ be the policy constructed as follows: (a) $\pi_L^i(\theta) = 0$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_L^i(\theta) = \pi(\theta)$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \in [0,1]$, $U_\sigma^{\Gamma_L^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.*

Proof of Result S3-A-1. Note that, under the new policy, $\pi_L^i(\theta) = \pi(\theta) \times \mathbf{1}\{\theta > \underline{\theta}_i\}$. The posterior beliefs $\Lambda_\sigma^{\Gamma_L^i}(\cdot|x, 1)$ about θ of an agent with exogenous signal x and endogenous signal $s = 1$ under the new policy Γ_L^i thus dominate, in the FOSD sense, the analogous beliefs

[21] Recall that D^Γ is the partition of $(0, \theta^{MS}]$ induced by the policy Γ.

[22] For any $\hat{\theta} \in [0,1]$, $\Gamma^{\hat{\theta}} = (\{0,1\}, \pi^{\hat{\theta}})$ is the deterministic monotone policy with cut-off $\hat{\theta}$.

$\Lambda_\sigma^\Gamma(\cdot|x,1)$ under the original policy Γ.[23] The result then follows from the fact that, given any default threshold θ, the payoff from pledging when the fundamentals are equal to $\tilde{\theta}$ and default occurs if and only if $\tilde{\theta} \leq \theta$ is nondecreasing in $\tilde{\theta}$. *End of Proof of Result S3-A-1.*

Result S3-A-2. *Pick any policy* $\Gamma = \{\{0,1\},\pi\} \in \mathbb{G}_\sigma$. *Given the partition* $D^\Gamma \equiv \left\{d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right] : i = 1,...,N\right\}$ *of* $(0,\theta^{MS}]$ induced by Γ, *take any cell* $d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right]$, $i \geq 2$, *for which* $\pi(\theta) = 1$ *for all* $\theta \in d_i$. *Let* $\Gamma_R^i = \{\{0,1\},\pi_R^i\} \in \mathbb{G}_\sigma$ *be the policy constructed from* Γ *as follows: (a)* $\pi_R^i(\theta) = \pi(\theta)$ *for all* $\theta \leq \underline{\theta}_i$*; and (b)* $\pi_R^i(\theta) = 1$ *for all* $\theta > \underline{\theta}_i$. *Then, for all* $\theta \leq \bar{\theta}_i$, $U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) \geq U_\sigma^\Gamma(x_\sigma^*(\theta),1|x_\sigma^*(\theta))$.

Proof of Result S3-A-2. Let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ and $\Theta_i^0 \equiv \{\theta \in (\underline{\theta}_i, 1] : \pi(\theta) = 0\}$. For any $\theta^\# \leq \bar{\theta}_i$, and any x,

$$\begin{aligned}
\Lambda_\sigma^{\Gamma_R^i}\left(\theta^\#|x,1\right) &= Pr\{\theta \leq \theta^\#|x, \theta \in (\Theta^1 \cup \Theta_i^0)\} = \frac{Pr\{\theta \leq \theta^\# \wedge \theta \in (\Theta^1 \cup \Theta_i^0)|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} \\
&= \frac{Pr\{\theta \leq \theta^\# \wedge \theta \in \Theta^1|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} + \frac{Pr\{\theta \leq \theta^\# \wedge \theta \in \Theta_i^0|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} = \frac{Pr\{\theta \leq \theta^\# \wedge \theta \in \Theta^1|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} \\
&\leq Pr\{\theta \leq \theta^\#|x, \theta \in \Theta^1\} = \Lambda_\sigma^\Gamma(\theta^\#|x,1).
\end{aligned}$$

Given the above inequality, and the fact that $b < 0 < g$, we then have that, for any $\theta \leq \bar{\theta}_i$,

$$\begin{aligned}
U_\sigma^\Gamma(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) &= b \cdot \Lambda_\sigma^\Gamma(\theta|x_\sigma^*(\theta),1) + g \cdot \left(1 - \Lambda_\sigma^\Gamma(\theta|x_\sigma^*(\theta),1)\right) \\
&\leq b \cdot \Lambda_\sigma^{\Gamma_R^i}(\theta|x_\sigma^*(\theta),1) + g \cdot \left(1 - \Lambda_\sigma^{\Gamma_R^i}(\theta|x_\sigma^*(\theta),1)\right) = U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)).
\end{aligned}$$

End of Proof of Result S3-A-2.

Property S3-A follows from Results S3-A-1 and S3-A-2, by taking the cell $d_i = (\theta', \theta'']$. □

Now, fix $\varepsilon \in (0,\theta^{MS})$. For any $\theta^* \in [0,\theta^{MS}-\varepsilon]$, let Γ^{θ^*} be the monotone rule with cut-off θ^*. For any $\theta^* \in [0,\theta^{MS}-\varepsilon]$, any $\sigma \in \mathbb{R}_{++}$, let

$$H_\sigma(\theta^*;\varepsilon) \equiv \inf_{\theta \in [\theta^*,\theta^*+\varepsilon]} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)).$$

Note that $U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta),1|x_\sigma^*(\theta))$ is continuous in (θ^*,θ,σ) over $[0,1]^2 \times (0,\hat{\sigma}]$. From Berge's Maximum Theorem, $H_\sigma(\theta^*;\varepsilon)$ is thus continuous in (θ^*,σ) over $[0,\theta^{MS}-\varepsilon] \times (0,\hat{\sigma}]$.

For all $\theta^* \in [0,\theta^{MS}-\varepsilon]$, all $\theta \in (\theta^*,\theta^*+\varepsilon]$, $\lim_{\sigma \to 0^+} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) = \int_0^1 u(\theta,A)\mathrm{d}A$. Because $\int_0^1 u(\theta,A)\mathrm{d}A$ is strictly increasing in θ and equal to zero at $\theta = \theta^{MS}$, for any $\theta^* \in [0,\theta^{MS}-\varepsilon]$, $H_{0^+}(\theta^*;\varepsilon) \equiv \lim_{\sigma\to 0^+} H_\sigma(\theta^*;\varepsilon) = \lim_{\sigma\to 0^+}\lim_{\theta\to\theta^{*+}} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) = \int_0^1 u(\theta^*,A)\mathrm{d}A$. We show next that $H_\sigma(\cdot;\varepsilon)$ converges uniformly to the limit function $H_{0^+}(\cdot;\varepsilon)$ over $[0,\theta^{MS}-\varepsilon]$.

[23] No matter the shape of the beliefs $\Lambda_\sigma^\Gamma(\cdot|x,1)$, the announcement that $\theta > \underline{\theta}_i$ is always "good news" in the sense of Milgrom (1981) and hence $\Lambda_\sigma^{\Gamma_L^i}(\cdot|x,1) \succ_{FOSD} \Lambda_\sigma^\Gamma(\cdot|x,1)$.

Property S3-B. *Fix $\varepsilon \in (0, \theta^{MS})$. For any $\epsilon < \varepsilon$, there exists $\sigma'(\epsilon) > 0$ such that, for any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $|H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon)| < \epsilon$.*

Proof of Property S3-B. The limit function $H_{0^+}(\cdot; \varepsilon)$ is *uniformly* continuous over $[0, \theta^{MS} - \varepsilon]$. As a consequence, there exists $\delta > 0$ such that for any $\theta, \tilde{\theta} \in [0, \theta^{MS} - \varepsilon]$, with $|\tilde{\theta} - \theta| \leq \delta$, necessarily $|H_{0^+}(\tilde{\theta}; \varepsilon) - H_{0^+}(\theta; \varepsilon)| < \epsilon/2$. Next, let $D_\delta \equiv \{(\underline{\theta}_i, \bar{\theta}_i] : i = 1, ..., N\}$, $N \in \mathbb{N}$, be any interval partition of $(0, \theta^{MS} - \varepsilon]$ with the property that every cell $(\underline{\theta}_i, \bar{\theta}_i] \in D_\delta$ is such that $|\bar{\theta}_i - \underline{\theta}_i| \leq \delta$. For any $i = 1, ..., N$, any $\sigma > 0$, let $\hat{\theta}^i_\sigma \equiv \sup\{\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)\}$. That $H_\sigma(\theta; \varepsilon)$ is continuous in (σ, θ) implies that the hypothesis of Berge's Maximum Theorem hold and, hence, the correspondence $\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)$ is compact-valued and upper hemi-continuous in σ. As a result, for any $\sigma > 0$, $\hat{\theta}^i_\sigma = \max\{\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)\}$. Moreover, $\lim_{\sigma \to 0^+} H_\sigma(\hat{\theta}^i_\sigma; \varepsilon) = H_{0^+}(\hat{\theta}^i_{0^+}; \varepsilon)$, where $\hat{\theta}^i_{0^+} \equiv \lim_{\sigma \to 0^+} \hat{\theta}^i_\sigma$.

For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let $(\underline{\theta}_j, \bar{\theta}_j] \in D_\delta$ be the partition cell containing θ^*. Then,

$$\begin{gathered} H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) \leq H_\sigma(\hat{\theta}^j_\sigma; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) \\ = H_\sigma(\hat{\theta}^j_\sigma; \varepsilon) - H_{0^+}(\hat{\theta}^j_{0^+}; \varepsilon) + H_{0^+}(\hat{\theta}^j_{0^+}; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) < H_\sigma(\hat{\theta}^j_\sigma; \varepsilon) - H_{0^+}(\hat{\theta}^j_{0^+}; \varepsilon) + \epsilon/2 < \epsilon \end{gathered}$$

for all $\sigma < \bar{\sigma}_j(\epsilon)$, for some $\bar{\sigma}_j(\epsilon) > 0$. The first inequality is by definition of $\hat{\theta}^i_\sigma$. The second inequality follows from the fact that $|\hat{\theta}^j_{0^+} - \theta^*| < \delta$. The last inequality follows from the fact that $\lim_{\sigma \to 0^+} H_\sigma(\hat{\theta}^j_\sigma) = H_{0^+}(\hat{\theta}^j_{0^+})$. Similar arguments imply that $H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) > -\epsilon$ for all $\sigma < \underline{\sigma}_j(\epsilon)$, for some $\underline{\sigma}_j(\epsilon) > 0$.

Now let $\sigma'(\epsilon) \equiv \min\{\min_{i \in N}\{\bar{\sigma}_i(\epsilon)\}, \min_{i \in N}\{\underline{\sigma}_i(\epsilon)\}\}$. For any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, we thus have that $|H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon)| < \epsilon$, thus proving that $H_\sigma(\cdot; \varepsilon)$ converges uniformly to $H_{0^+}(\cdot; \varepsilon)$ as $\sigma \to 0^+$. This completes the proof of Property S3-B. □

Next, given $\varepsilon \in (0, \theta^{MS})$, pick an arbitrary $\eta \in (\int_0^1 u(\theta^{MS} - \varepsilon, A)\mathrm{d}A, 0)$. Because $H_{0^+}(\theta^*; \varepsilon) \leq \eta$ for all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, and because $H_\sigma(\cdot; \varepsilon)$ converges uniformly to $H_{0^+}(\cdot; \varepsilon)$, there exists $\sigma(\varepsilon) > 0$ such that, for any $\sigma < (\varepsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_\sigma(\theta^*; \varepsilon) \leq \eta < 0$. Therefore, for any $\sigma < \sigma(\varepsilon)$, and any monotone policy Γ^{θ^*} with cut-off $\theta^* \in [0, \theta^{MS} - \varepsilon]$, there exists $\theta \in [\theta^*, \theta^* + \varepsilon]$ such that $U^{\Gamma^{\theta^*}}_\sigma(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)) \leq \eta$.

Together, Properties S3-A and S3-B then imply that, for any $\sigma < \sigma(\varepsilon)$, and any policy Γ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$ for some $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $U^\Gamma_\sigma(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)) < 0$ for some $\theta \in (\theta', \theta'']$. Hence $\Gamma \notin \mathbb{G}_\sigma$. The claim in Lemma S3-A then follows by contrapositive. This completes the proof of Lemma S3-A. ■

Step 2. Next, we show that, for any policy $\Gamma = (\{0, 1\}, \pi) \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (0, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^\# \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma^\#) < M(\Gamma)$ such that, when agents play as in G_σ, the probability of default under $\Gamma^\#$ is strictly smaller than under Γ. The result follows from Lemmas S3-B, S3-C and

S3-D below.

Lemma S3-B. *For any* $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ *such that* $\inf_{\theta\in[0,1]} \; U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) > 0$, *there exists another policy* $\hat{\Gamma} = (\{0,1\}, \hat{\pi}) \in \mathbb{G}_\sigma$*, with* $M(\hat{\Gamma}) \le M(\Gamma)$*, such that, in the auxiliary game* G_σ*, the probability of default under* $\hat{\Gamma}$ *is strictly smaller than under* Γ*.*

Proof of Lemma S3-B. That $\inf_{\theta\in[0,1]} \; U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) > 0$ implies that, starting from $\Gamma = (\{0,1\}, \pi)$, one can construct another policy $\hat{\Gamma} = (\{0,1\}, \hat{\pi})$ sufficiently close to Γ (in the L_1 norm) and such that $\hat{\pi}(\theta) \ge \pi(\theta)$ for all θ, with the inequality strict over some positive F-measure set $(\tilde{\theta}', \tilde{\theta}'') \subseteq (0,1]$, and such that (a) $\hat{\pi}(\theta) = 0$ for all $\theta \le 0$, (b) $\hat{\pi}(\theta) = 1$ for all $\theta > 1$, (c) $U_\sigma^{\hat{\Gamma}}(x, 1|x) \ge 0$ all x, and (d) $M(\hat{\Gamma}) \le M(\Gamma)$. By definition of $\mathbb{G}_\sigma$, $\hat{\Gamma} \in \mathbb{G}_\sigma$. That, in the auxiliary game G_σ, the probability of default under $\hat{\Gamma}$ is strictly smaller than under Γ, then follows from the fact that all agents pledge when they receive the signal $s = 1$. This completes the proof of Lemma S3-B. ■

For any $\sigma > 0$, and any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, $U_\sigma^\Gamma\left(x_\sigma^*\left(\cdot\right), 1|x_\sigma^*\left(\cdot\right)\right)$ is continuous over $[0,1]$. Hence $\inf_{\theta\in[0,1]} \; U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) = \min_{\theta\in[0,1]} \; U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right)$.

Lemma S3-C. *Let* $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ *be such that* $min_{\theta\in[0,1]} \; U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) = 0$*. For any* $\theta_\sigma^\sharp \in \arg min_{\theta\in[0,1]} \; U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right)$*, there exists* $\gamma_\sigma^\Gamma > 0$ *such that* $\pi\left(\theta\right) = 1$ *for* $F-$*almost all* $\theta \in \left(\theta_\sigma^\sharp - \gamma_\sigma^\Gamma, \theta_\sigma^\sharp\right)$*.*

Proof of Lemma S3-C. The proof is by contraposition. Suppose there exists $\delta > 0$ such that $\pi\left(\theta\right) = 0$ for F-almost all $\theta \in \left(\theta_\sigma^\sharp - \delta, \theta_\sigma^\sharp\right)$. Observe that the sign of

$$U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right)\right)$$

is the same as the sign of

$$b\int_{-\infty}^{\theta_\sigma^\sharp - \delta} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) + g\int_{\theta_\sigma^\sharp - \delta}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta).$$

Next observe that

$$\begin{aligned}
0 &= U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp\right)\right)\int_{-\infty}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&= \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta \le \theta_\sigma^\sharp\right\} + g\mathbf{1}\left\{\theta > \theta_\sigma^\sharp\right\}\right)\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&> \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta \le \theta_\sigma^\sharp\right\} + g\mathbf{1}\left\{\theta > \theta_\sigma^\sharp\right\}\right)\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&= \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta \le \theta_\sigma^\sharp - \delta\right\} + g\mathbf{1}\left\{\theta > \theta_\sigma^\sharp - \delta\right\}\right)\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&= U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right)\right)\int_{-\infty}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)
\end{aligned}$$

The first equality follows from the assumptions of the lemma. The second equality follows from the definition of the function $U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp\right)\right)$. The inequality follows from the monotonicity of $x^*_\sigma(\cdot)$, the fact that $\phi((x-\theta)/\sigma)$ is log-supermodular in (x,θ), and Property SCB in the proof of Theorem 2 in the main text. The third equality follows from the fact that $\pi(\theta)=0$ for F-almost all $\theta \in \left(\theta^\sharp_\sigma-\delta, \theta^\sharp_\sigma\right)$. The last equality follows from the definition of the function $U^\Gamma_\sigma(x^*_\sigma(\theta^\sharp_\sigma-\delta),1|x^*_\sigma(\theta^\sharp_\sigma-\delta))$. Hence, $U^\Gamma_\sigma(x^*_\sigma(\theta^\sharp_\sigma-\delta),1|x^*_\sigma(\theta^\sharp_\sigma-\delta))<0$, thus contradicting the assumption that $\Gamma \in \mathbb{G}_\sigma$. This completes the proof of Lemma S3-C. ■

Lemma S3-D. *For any $\varepsilon>0$, there exists $\sigma^\#(\varepsilon)\in(0,\hat{\sigma})$ such that, for any $\sigma\in(0,\sigma^\#(\varepsilon)]$, and any policy $\Gamma=(\{0,1\},\pi)\in\mathbb{G}_\sigma$ for which there exists $(\theta',\theta'']\in D^\Gamma$ such that (a) $|\theta''-\theta'|>\varepsilon$ and (b) $\pi(\theta)=0$ for all $\theta\in(\theta',\theta'']$, there exists another policy $\Gamma^\#=\left(\{0,1\},\pi^\#\right)\in\mathbb{G}_\sigma$, with $M(\Gamma^\#)\le M(\Gamma)$, such that, in the auxiliary game G_σ, the probability of default under $\Gamma^\#$ is strictly smaller than under Γ.*

Proof of Lemma S3-D. For any $\theta\in(0,1)$, $\lim_{\sigma\to 0^+} x^*_\sigma(\theta)\equiv x^*_{0^+}(\theta)=\theta$. Furthermore, for any $\varepsilon\in(0,\min\{\theta^{MS},1-\theta^{MS}\})$, the function $x^*_{0^+}:\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]\to\mathbb{R}$ is uniformly continuous. Hence, for any $\delta<\varepsilon/4$, there exists $\tilde{\sigma}(\delta)>0$ such that, for any $\sigma\in(0,\tilde{\sigma}(\delta)]$, and any $\theta\in\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]$, we have that $|x^*_\sigma(\theta)-\theta|\le\delta$.[24] In turn, this implies that, for any $\varepsilon>0$ small, there exists $\sigma^\#(\varepsilon)\in(0,\hat{\sigma}]$ such that, for any $\sigma\in(0,\sigma^\#(\varepsilon)]$, and any $(\theta',\theta'']\in D^\Gamma$ such that $|\theta''-\theta'|>\varepsilon$, we have that, for any $\theta\in[\theta'',1-\frac{\varepsilon}{4}]$, $|\theta-x^*_\sigma(\theta)|<|(\theta'+\theta'')/2-x^*_\sigma(\theta)|$. Likewise, for any $\theta\in[\varepsilon/4,\theta']$, and any $\hat{\theta}\ge\theta''$, we have that $|\theta-x^*_\sigma(\theta)|<|x^*_\sigma(\theta)-\hat{\theta}|$ when $\sigma\in(0,\sigma^\#(\varepsilon)]$.

Next, pick any policy $\Gamma=(\{0,1\},\pi)\in\mathbb{G}_\sigma$ for which there exists $d\equiv(\theta',\theta'']\in D^\Gamma$ such that (a) $|\theta''-\theta'|>\varepsilon$ and (b) $\pi(\theta)=0$ for all $\theta\in(\theta',\theta'']$. If $\min_{\theta\in[0,1]}U^\Gamma_\sigma(x^*_\sigma(\theta),1|x^*_\sigma(\theta))>0$, the result follows directly from Lemma S3-B. Thus assume that $\min_{\theta\in[0,1]}U^\Gamma_\sigma(x^*_\sigma(\theta),1|x^*_\sigma(\theta))=0$.

Suppose that $\min_{\theta\in[\theta'',1]}U^\Gamma_\sigma(x^*_\sigma(\theta),0|x^*_\sigma(\theta))>0$. By Lemma S3-C, $U^\Gamma_\sigma(x^*_\sigma(\theta),1|x^*_\sigma(\theta))>0$ for all $\theta\in(\theta',\theta'']$. Hence, $\min_{\theta\in[\theta',1]}U^\Gamma_\sigma(x^*_\sigma(\theta),1|x^*_\sigma(\theta))>0$.

Below we show that, starting from Γ, we can then construct a policy $\Gamma^\eta\in\mathbb{G}_\sigma$, with $M(\Gamma^\eta)\le M(\Gamma)$ such that, when agents play as in G_σ, the probability of default under Γ^η is strictly smaller than under Γ. Γ^η is obtained from Γ by giving a pass grade to a positive-measure interval of types in the middle of $(\theta',\theta'']$. Formally, take $\eta\in(0,(\theta''-\theta')/2)$ and let $\Gamma^\eta=(\{0,1\},\pi^\eta)$ be the policy whose rule π^η is given by (a) $\pi^\eta(\theta)=\pi(\theta)$ for all $\theta\notin[(\theta'+\theta'')/2,(\theta'+\theta'')/2+\eta]$, and (b) $\pi^\eta(\theta)=1$ for all $\theta\in[(\theta'+\theta'')/2,(\theta'+\theta'')/2+\eta]$. Below we show that $U^{\Gamma^\eta}(x^*_\sigma(\theta),1|x^*_\sigma(\theta))\ge 0$ for all $\theta\in[0,1]$. To see this, let $\Theta^1\equiv\{\theta\in\Theta:\pi(\theta)=1\}$ be the collection of fundamentals receiving a pass grade under the original policy Γ. For any

[24] The proof for the existence of a sequence $\left\{x^*_{\sigma_n}(\cdot)\right\}_n$ with domain $\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]$ converging uniformly to its limit function $x^*_{0^+}(\cdot)$ follows from the same arguments that establish the uniform convergence of $\{H_{\sigma_n}(\cdot)\}_n$ to $H_{0^+}(\cdot)$ in Step 1.

$\theta \in [0, \theta']$, and any x,

$$\Lambda_\sigma^{\Gamma^\eta}(\theta|x,1) = Pr\{\tilde{\theta} \leq \theta | x, \tilde{\theta} \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])\}$$
$$= \frac{Pr\{\tilde{\theta} \leq \theta \wedge \tilde{\theta} \in \Theta^1 | x\}}{Pr\{\tilde{\theta} \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])|x\}} \leq Pr\{\tilde{\theta} \leq \theta | x, \tilde{\theta} \in \Theta^1\} = \Lambda_\sigma^\Gamma(\theta|x,1).$$

The first equality follows from the fact that, under Γ^η, the signal $s = 1$ carries the same information as the announcement that $\tilde{\theta} \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])$. The inequality follows from the fact that $Pr\left\{\tilde{\theta} \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])\,|x\right\} > Pr\left\{\tilde{\theta} \in \Theta^1 | x\right\}$. The last equality follows from fact that, under the original policy Γ, the signal $s = 1$ carries the same information as the announcement that $\tilde{\theta} \in \Theta^1$.

Given the above inequality, and the fact that, $b < 0 < g$, we then have that, for any $\theta \in [0, \theta']$,

$$U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = b \cdot \Lambda_\sigma^\Gamma(\theta|x_\sigma^*(\theta), 1) + g \cdot \left[1 - \Lambda_\sigma^\Gamma(\theta|x_\sigma^*(\theta), 1)\right]$$
$$\leq b \cdot \Lambda_\sigma^{\Gamma^\eta}(\theta|x_\sigma^*(\theta), 1) + g \cdot \left[1 - \Lambda_\sigma^{\Gamma^\eta}(\theta|x_\sigma^*(\theta), 1)\right] = U_\sigma^{\Gamma^\eta}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).$$

Hence $U^{\Gamma^\eta}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq 0$, all $\theta \leq \theta'$. That $\min_{\theta \in [\theta', 1]} U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$, along with the continuity of $U_\sigma^{\Gamma^\eta}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ in η implies that $\min_{\theta \in [0,1]} U_\sigma^{\Gamma^\eta}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq 0$ for η small. Hence $\Gamma^\eta \in \mathbb{G}_\sigma$.

Next, consider the more interesting case in which $\min_{\theta \in [\theta'', 1]} U_\sigma^\Gamma(x_\sigma^*(\theta), 0|x_\sigma^*(\theta)) = 0$. Let $\theta_\sigma^\# \equiv \inf\left\{\theta \geq \theta'' : U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = 0\right\}$. An implication of Lemma S3-C is that that $\theta_\sigma^\# > \theta''$. Also let $(\theta''', \theta''''] \subset [0,1]$ be the first interval to the immediate right of $(\theta', \theta'']$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta''', \theta'''']$ and let $\hat{\theta} = \min\left\{\theta'''', \theta_\sigma^\#\right\}$.[25]

Now, pick $\xi > 0$ small and let $\delta(\xi)$ be implicitly defined by

$$F((\theta'+\theta'')/2+\xi) - F((\theta'+\theta'')/2) = F((\theta'''+\hat{\theta})/2+\delta(\xi)) - F((\theta'''+\hat{\theta})/2). \qquad \text{(S7)}$$

Consider the policy $\Gamma^\xi = (\{0,1\}, \pi^\xi)$ defined by (a) $\pi^\xi(\theta) = \pi(\theta)$ for all $\theta \notin [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\xi] \cup [(\theta'''+\hat{\theta})/2, (\theta'''+\hat{\theta})/2+\delta(\xi)]$, (b) $\pi^\xi(\theta) = 1$ for all $\theta \in [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\xi]$, and (c) $\pi^\xi(\theta) = 0$ for all $\theta \in [(\theta'''+\hat{\theta})/2, (\theta'''+\hat{\theta})/2+\delta(\xi)]$. Below we establish that, when $\xi > 0$ is small, such a policy is such that $\min_{\theta \in [0,1]} U^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$ and hence $\Gamma^\xi \in \mathbb{G}_\sigma$. To see this, for any arbitrary policy $\tilde{\Gamma} = \{\{0,1\}, \tilde{\pi}\}$, any $\theta \in [0,1]$, let

$$V_\sigma^{\tilde{\Gamma}}(\theta) \equiv U_\sigma^{\tilde{\Gamma}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))\, p_\sigma^{\tilde{\Gamma}}(x_\sigma^*(\theta), 1),$$

where, for any x, $p_\sigma^{\tilde{\Gamma}}(x,1) \equiv \int_\Theta \tilde{\pi}(\theta) p_\sigma(x|\theta) \mathrm{d}F(\theta)$, with $p_\sigma(x|\theta) \equiv \frac{1}{\sigma}\phi((x-\theta)/\sigma)$.

[25] The existence of such an interval follows from the fact that $\pi(\theta) = 1$ in a left neighborhood of $\theta_\sigma^\#$ by virtue of Lemma S3-C. Also observe that, when $\theta'' < \theta^{MS}$, such an interval is adjacent to $(\theta', \theta'']$ and hence $\theta''' = \theta''$.

By definition of $\theta_\sigma^\#$, we must have that, for all θ, $0 = V_\sigma^\Gamma\left(\theta_\sigma^\#\right) \leq V_\sigma^\Gamma(\theta)$. Next, for any $\xi > 0$, define $\varphi_R(\xi) \equiv \min_{\theta\in[\theta'',1]} V_\sigma^{\Gamma^\xi}(\theta)$. Let $\overline{u}(\tilde\theta,\theta) \equiv g\mathbf{1}\{\tilde\theta > \theta\} + b\mathbf{1}\{\tilde\theta \leq \theta\}$ and note that, for any θ,

$$V_\sigma^{\Gamma^\xi}(\theta) = V_\sigma^\Gamma(\theta) + \int_{(\theta'+\theta'')/2}^{(\theta'+\theta'')/2+\xi} \overline{u}(\tilde\theta,\theta) p_\sigma(x_\sigma^*(\theta)\,|\tilde\theta)\mathrm{d}F(\tilde\theta) - \int_{(\theta'''+\hat\theta)/2}^{(\theta'''+\hat\theta)/2+\delta(\xi)} \overline{u}(\tilde\theta,\theta) p_\sigma(x_\sigma^*(\theta)\,|\tilde\theta)\mathrm{d}F(\tilde\theta).$$

Using the envelope theorem, we have that, for any $\theta_\sigma^\xi \in \arg\min_{\theta\in[\theta'',1]} V_\sigma^{\Gamma^\xi}(\theta)$,

$$\begin{aligned}
\varphi_R'(\xi) &= f\left((\theta'+\theta'')/2+\xi\right)\overline{u}\left((\theta'+\theta'')/2+\xi,\theta_\sigma^\xi\right)p_\sigma\left(x_\sigma^*(\theta_\sigma^\xi)|(\theta'+\theta'')/2+\xi\right) \\
&\quad - f\left((\theta'''+\hat\theta)/2+\delta(\xi)\right)\overline{u}((\theta'''+\hat\theta)/2+\delta(\xi),\theta_\sigma^\xi)p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta'''+\hat\theta)/2+\delta(\xi))\delta'(\xi) \\
&= f\left((\theta'+\theta'')/2+\xi\right)[\overline{u}((\theta'+\theta'')/2+\xi,\theta_\sigma^\xi)p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta'+\theta'')/2+\xi) \\
&\quad - \overline{u}((\theta'''+\hat\theta)/2+\delta(\xi),\theta_\sigma^\xi)p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta'''+\hat\theta)/2+\delta(\xi))],
\end{aligned}$$

where the second equality uses the implicit function theorem applied to (S7) to obtain that $\delta'(\xi) = f\left((\theta'+\theta'')/2+\xi\right)/f((\theta'''+\hat\theta)/2+\delta(\xi))$. As a consequence,

$$\begin{aligned}
\lim_{\xi\to0^+}\varphi_R'(\xi) = f\left((\theta'+\theta'')/2\right)[\overline{u}\left((\theta'+\theta'')/2,\theta_\sigma^\#\right)p_\sigma\left(x_\sigma^*(\theta_\sigma^\#)|(\theta'+\theta'')/2\right) \\
- \overline{u}((\theta'''+\hat\theta)/2,\theta_\sigma^\#)p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta'''+\hat\theta)/2)].
\end{aligned} \tag{S8}$$

That $\sigma < \sigma^\#(\varepsilon)$ implies that $|x_\sigma^*\left(\theta_\sigma^\#\right) - (\theta'''+\hat\theta)/2| < |x_\sigma^*\left(\theta_\sigma^\#\right) - (\theta'+\theta'')/2|$. That $p_\sigma(x|\theta)$ is single-peaked in turn implies that $p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta'+\theta'')/2) < p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta'''+\hat\theta)/2)$ and hence that

$$\begin{aligned}
\overline{u}\left((\theta'+\theta'')/2,\theta_\sigma^\#\right)p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta'+\theta'')/2\right) - \overline{u}\left((\theta'''+\hat\theta)/2,\theta_\sigma^\#\right)p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta'''+\hat\theta)/2\right) \\
= b\times\left(p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta'+\theta'')/2\right) - p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta'''+\hat\theta)/2\right)\right) > 0.
\end{aligned}$$

Thus, $\lim_{\xi\to0^+}\varphi_R'(\xi) > 0$. By continuity of $U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta),1|x_\sigma^*(\theta))$ in ξ, we then have that, for $\xi > 0$ small, $\min_{\theta\in[\theta'',1]} U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) > 0$.

Next, we prove that, under the policy Γ^ξ, $\min_{\theta\in[0,\theta'']} U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) > 0$. For any $\xi > 0$, define $\varphi_L(\xi) \equiv \min_{\theta\in[0,\theta']} V_\sigma^{\Gamma^\xi}(\theta)$. Arguments similar to those used above to compute $\lim_{\xi\to0^+}\varphi_R'(\xi)$ imply that, for any $\theta_\sigma^{\#\#} \in \arg\min_{\theta\in[0,\theta']} V_\sigma^\Gamma(\theta)$, when $\sigma \leq \sigma^\#(\varepsilon)$,

$$\begin{aligned}
\lim_{\xi\to0^+}\varphi_L'(\xi) &= f\left((\theta'+\theta'')/2\right)[\overline{u}((\theta'+\theta'')/2,\theta_\sigma^{\#\#})p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta'+\theta'')/2) \\
&\quad - \overline{u}((\theta'''+\hat\theta)/2,\theta_\sigma^{\#\#})p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta'''+\hat\theta)/2)] \\
&= f\left((\theta'+\theta'')/2\right)g\left[p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta'+\theta'')/2) - p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta'''+\hat\theta)/2))\right] > 0.
\end{aligned}$$

The first equality follows from steps analogous to those used to establish (S8). The second equality follows from the fact that, by assumption $\theta_\sigma^{\#\#} \leq \theta'$. The inequality is a consequence of the fact that, for $\sigma \leq \sigma^{\#}(\varepsilon)$, $|x^*_\sigma\left(\theta^{\#\#}_\sigma\right) - (\theta' + \theta'')/2| < |x^*_\sigma\left(\theta^{\#\#}_\sigma\right) - (\theta''' + \hat{\theta})/2)|$, which, together with the fact that the noise distribution is single-peaked, implies that

$$p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta' + \theta'')/2) > p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta''' + \hat{\theta})/2).$$

Hence, for $\xi > 0$ small, $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$ for all $\theta \in [0, \theta']$. Furthermore, by Lemma S3-C, $U^{\Gamma}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, provided that ξ is small, the continuity of $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right)$ in ξ implies that $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$ also for $\theta \in (\theta', \theta'']$. Combining all the properties above, we thus conclude that, for $\xi > 0$ small, $\min_{\theta\in[0,1]} U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$. Hence $\Gamma^\xi \in \mathbb{G}_\sigma$.

By construction, $M(\Gamma^\xi) < M(\Gamma)$. Furthermore, when agents play according to G_σ, the probability of default under Γ^ξ is the same as under Γ. Lemma S3-B then implies that, starting from Γ^ξ, one can construct a policy $\Gamma^{\#} \in \mathbb{G}_\sigma$, close to Γ^ξ in the L_1 norm, such that (1) $M(\Gamma^{\#}) \leq M(\Gamma^\xi)$ and (2), when agents play according to G_σ, the probability of default under $\Gamma^{\#}$ is strictly smaller than under Γ. This completes the proof of Lemma S3-D. ■

Step 3. Steps 1 and 2 imply that there exists a function $\bar{\sigma} : \left(0, \min\{\theta^{MS}, 1-\theta^{MS}\}\right) \to \mathbb{R}_{++}$, with $\bar{\sigma}(\varepsilon) \leq \min\{\sigma(\varepsilon), \sigma^{\#}(\varepsilon)\}$ for all $\varepsilon \in \left(0, \min\{\theta^{MS}, 1-\theta^{MS}\}\right)$ and with $\bar{\sigma}(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$, such that the following is true: For any $\varepsilon \in \left(0, \min\{\theta^{MS}, 1-\theta^{MS}\}\right)$, any $\sigma \in (0, \bar{\sigma}(\varepsilon)]$, and any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ with $M(\Gamma) > \varepsilon$, there exists another policy $\Gamma' = (\{0,1\}, \pi') \in \mathbb{G}_\sigma$ with $M(\Gamma') \leq \varepsilon$ such that, when the agents play as in the auxiliary game G_σ, the probability of default under Γ' is strictly smaller than under Γ.[26]

Furthermore, the arguments establishing Lemma S3-D reveal that the policy Γ' can be constructed so that $U^{\Gamma'}_\sigma(x, 1|x) > 0$ for all x. The policy Γ' thus satisfies PCP also when agents play according to MARP. The claim in the Example then follows by taking $\Gamma^* = \Gamma'$ with Γ' satisfying the above properties.

Step 4. We now complete the proof by showing how to construct the function $\mathcal{E}$ in the example. Let (ε_n) be a non-increasing sequence satisfying $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$, then let $\sigma_n = \overline{\sigma}(\varepsilon_n)$, with the function $\overline{\sigma}(\cdot)$ as defined in Step 3. The results in Steps 1-3 above imply that, given $(\varepsilon_n, \sigma_n)$, there exist strictly decreasing subsequences $(\tilde{\varepsilon}_n)$ and $(\tilde{\sigma}_n)$ satisfying $\lim_{n\to\infty} \tilde{\varepsilon}_n = \lim_{n\to\infty} \tilde{\sigma}_n = 0$ such that, for any $n \in \mathbb{N}$, the conclusions in Step 3 hold for $\varepsilon = \tilde{\varepsilon}_n$ and $\bar{\sigma}(\varepsilon_n) = \tilde{\sigma}_n$. Then let $\bar{\sigma} = \tilde{\sigma}_0 > 0$ and $\mathcal{E} : (0, \bar{\sigma}] \to \mathbb{R}_+$ be the function defined by $\mathcal{E}(\sigma) = \varepsilon_n$ for all $\sigma \in (\sigma_{n+1}, \sigma_n]$. The result in the example then follows from Steps 1-3, by letting $\mathcal{E}(\cdot)$ be the function so constructed. Q.E.D.

[26] Observe that the thresholds $\sigma(\varepsilon)$ and $\sigma^{\#}(\varepsilon)$ identified in Steps 1 and 2 above are invariant to the initial policy Γ. The same arguments used to arrive at a policy $\Gamma^{\#}$ with mesh $M(\Gamma^{\#}) < M(\Gamma)$ can then be iterated till one arrives at a policy Γ' with mesh $M(\Gamma') \leq \varepsilon$.

Section S4: Extended Model

Enrichments. The fundamentals are given by (θ, z), with θ drawn from Θ according to F, and with z drawn from $[\underline{z}, \bar{z}]$ according to $Q_\theta(z)$, with the cdf $Q_\theta(z)$ weakly decreasing in θ, for any z.[27] The variable θ continues to parametrize the maximal information the policy maker can collect about the fundamentals. Likewise, any information the agents possess about z is encoded in the signals x they receive about θ.[28] The variable z proxies for macroeconomic variables that are only imperfectly correlated with the bank's fundamentals, and/or the exogenous supply of funds from sources other than the agents under consideration.

There exists a function $R : \Theta \times [0, 1] \times [\underline{z}, \bar{z}] \to \mathbb{R}$ such that, given (θ, A, z), default occurs (i.e., $r = 0$) if, and only if, $R(\theta, A, z) \leq 0$. The function R is continuous, strictly increasing in (θ, z, A), and such that $R(\underline{\theta}, 1, \overline{z}) = R(\bar{\theta}, 0, \underline{z}) = 0$, for some $\underline{\theta}, \bar{\theta} \in \mathbb{R}$, with $\underline{\theta} < \bar{\theta}$. The thresholds $\underline{\theta}$ and $\bar{\theta}$ define the "critical region" $(\underline{\theta}, \bar{\theta}]$ where the fate of the bank depends on the response of the market. For any (θ, A), the probability the bank avoids default is thus given by $r(\theta, A) \equiv \Pr\{R(\theta, A, z) > 0|\theta, A\}$.

The policy maker's payoff is

$$\hat{U}^P(\theta, A, z) = \hat{W}(\theta, A, z)\mathbf{1}\{R(\theta, A, z) > 0\} + \hat{L}(\theta, A, z)\mathbf{1}\{R(\theta, A, z) \leq 0\}.$$

whereas the agents' payoff differential between the "friendly" and the "adversarial" action is

$$\hat{u}(\theta, A, z) = \hat{g}(\theta, A, z)\mathbf{1}\{R(\theta, A, z) > 0\} + \hat{g}(\theta, A, z)\mathbf{1}\{R(\theta, A, z) \leq 0\}$$

with $\hat{g}(\theta, A, z) > 0 > \hat{b}(\theta, A, z)$, for any (θ, A, z). For any (θ, A), then let

$$g(\theta, A) \equiv \frac{\mathbb{E}\{\mathbf{1}(R(\theta, A, z) > 0)\hat{g}(\theta, A, z)|\theta, A\}}{r(\theta, A)} \text{ and } b(\theta, A) \equiv \frac{\mathbb{E}\{\mathbf{1}(R(\theta, A, z) \leq 0)\hat{b}(\theta, A, z)|\theta, A\}}{1 - r(\theta, A)}$$

denote the agents' expected payoff differential, respectively, in case of no default and in case of default, and, likewise, let

$$W(\theta, A) \equiv \frac{\mathbb{E}\{\mathbf{1}(R(\theta, A, z) > 0)\hat{W}(\theta, A, z)|\theta, A\}}{r(\theta, A)} \text{ and } L(\theta, A) \equiv \frac{\mathbb{E}\{\mathbf{1}(R(\theta, A, z) \leq 0)\hat{L}(\theta, A, z)|\theta, A\}}{1 - r(\theta, A)}$$

denote the policy maker's expected payoff, again in case of no default and default, respectively. The agents' and the policy maker's expected payoffs can then be conveniently expressed as a function of θ and A only, by letting

$$u(\theta, A) \equiv r(\theta, A)g(\theta, A) + (1 - r(\theta, A))b(\theta, A) \text{ and } U^P(\theta, A) \equiv r(\theta, A)W(\theta, A) + (1 - r(\theta, A))L(\theta, A).$$

[27] All the results extend to the case where $Q_\theta(z)$ has unbounded support.

[28] As in the baseline model, conditional on θ, the private signals $(x_i)_{i\in[0,1]}$ are i.i.d. draws from an (absolutely continuous) cumulative distribution function $P(x|\theta)$, with associated density $p(x|\theta)$ strictly positive over the interval $\varrho_\theta \in \mathbb{R}$.

Hereafter, we assume that both $u(\theta, A)$ and $U^P(\theta, A)$ are non-decreasing in A and that $U^P(\theta, 1) > U^P(\theta, 0)$ for all $\theta \in (\underline{\theta}, \bar{\theta}]$.[29]

Results. We now identify conditions under which Theorems 1-3 in the main text extend to these enriched economies.

Condition S4-FB. *For any x, $u(\theta, 1 - P(x|\theta)) \geq 0$ (alternatively, $u(\theta, 1 - P(x|\theta)) \leq 0$) implies that $u(\theta'', 1 - P(x|\theta'')) > 0$ for all $\theta'' > \theta$ (alternatively, $u(\theta', 1 - P(x|\theta')) < 0$ for all $\theta' < \theta$).*

Condition S4-FB requires that, for any x, $u(\theta, 1 - P(x|\theta))$ changes sign only once, from negative to positive. This property clearly holds when $u(\theta, A)$, in addition to being non-decreasing in A as assumed above, is also non-decreasing in θ. It also holds when the default outcome is a deterministic function of (θ, A), as in the baseline model, because, for any (θ, A), $g(\theta, A) > 0 > b(\theta, A)$, and $r(\theta, A)$ is non-decreasing in A.

Given any common posterior $G \in \Delta(\Theta)$, for any x such that $\int p(x|\theta)G(\mathrm{d}\theta) > 0$, let

$$\bar{U}^G(x) \equiv \left(\int u(\theta, 1 - P(x|\theta))p(x|\theta)G(\mathrm{d}\theta) \right) / \int p(x|\theta)G(\mathrm{d}\theta)$$

denote the expected payoff differential of an agent with signal x who expects all other agents to pledge if their private signal exceeds x and to not pledge otherwise. Let ξ^G be the largest solution to $\bar{U}^G(x) = 0$ if such an equation admits a solution, $\xi^G = +\infty$ if $\bar{U}^G(x) < 0$ for all x such that $\int p(x|\theta)G(\mathrm{d}\theta) > 0$, and $\xi^G = -\infty$ if $\bar{U}^G(x) > 0$ for all x such that $\int p(x|\theta)G(\mathrm{d}\theta)$. Finally, let $\theta^G \equiv \inf\left\{\theta : u(\theta, 1 - P(\xi^G|\theta)) \geq 0\right\}$. The interpretation of ξ^G and θ^G is the following. Suppose that the policy maker induces a common posterior G over Θ, $p(x|\theta)$ is log-supermodular, and Condition S4-FB holds. Then, in the continuation game that starts after the policy Γ induces the common posterior G, MARP is in cut-off strategies and is defined by the cut-off ξ^G.[30] When agents play according to MARP given the induced posterior G, their expected payoff differential is non-positive for all $\theta \leq \theta^G$ and non-negative for all $\theta > \theta^G$.

Condition S4-PC. *For any distribution $\Lambda \in \Delta(\Delta(\Theta))$ over posterior beliefs consistent with the common prior F (i.e., such that $\int G\Lambda(\mathrm{d}G) = F$), the following condition holds:*

$$\int \left(\int_{-\infty}^{\theta^G} U^P(\theta, 0)G(\mathrm{d}\theta) + \int_{\theta^G}^{+\infty} U^P(\theta, 1)G(\mathrm{d}\theta) \right) \Lambda(\mathrm{d}G) \geq \int \left(\int U^P(\theta, 1 - P(\xi^G|\theta))G(\mathrm{d}\theta) \right) \Lambda(\mathrm{d}G).$$

[29] That $u(\theta, A)$ is monotone in A implies that the continuation game remains supermodular. That $U^P(\theta, A)$ is non-decreasing in A implies that, for any Γ, MARP continues to coincide with the "smallest" rationalizable profile, that is, the one involving the smallest measure of agents pledging. Finally, that, for any θ in the critical region, the policy maker strictly prefers that all agents pledge to no agent pledging guarantees that, when the optimal policy has a pass/fail structure, it is obtained by maximizing the probability that a pass grade is given to banks whose fundamentals are in the critical range.

[30] The proof follows from arguments similar to those in the proof of Theorem 2 in the main text.

Condition S4-PC trivially holds when the policy maker faces no aggregate uncertainty (i.e., when each distribution Q_θ is degenerate), and W and L are invariant in A, as in the baseline model in Section 2 in the main text. More generally, Condition S4-PC accommodates for the possibility that both W and L depend on A, possibly non-monotonically, provided that, on average, the loss to the policy maker from having no agent pledge in states $\theta \leq \theta^G$ in which the agents' expected payoff differential (under MARP) is negative is more than compensated by the benefit from having all agents pledge in states $\theta > \theta^G$ in which the differential is positive. The average is over both the posteriors induced by the policy maker and the fundamentals. The condition thus requires that the policy maker's and the agents' payoffs be not too misaligned.

Theorem S4-1. *(a) Given any regular policy Γ, there exists a regular policy Γ^* satisfying* PCP *and such that, when agents play according to MARP, at any θ, their expected payoff differential under Γ^* is at least as high as under Γ.*[31] *Furthermore, when, under MARP, θ perfectly predicts the default outcome, the probability of default under Γ^* is the same as under Γ. (b) Suppose that $p(x|\theta)$ is log-supermodular and Condition S4-FB holds. The policy Γ^* from part (a) is a pass/fail policy. (c) If in addition to the conditions in part (b), Condition S4-PC also holds, then the policy maker's payoff under Γ^* is at least as high as under Γ. (d) Suppose that, in addition to the conditions in part (c), Condition M in the main text also holds. Then, Γ^* is a deterministic monotone policy.*

Proof of Theorem S4-1. The formal proof follows from arguments similar to those establishing Theorems 1-3 and is omitted for brevity.[32] Here we discuss the novel effects due to the enrichments introduced above and the role played by the conditions in the theorem.

First, consider part (a). When default depends on variables only imperfectly correlated with θ, perfect coordination cannot be induced by announcing to the agents the fate of the regime under MARP, as in the proof of Theorem 1 in the main text. Perfect coordination, however, can still be induced by announcing, at any θ, the *sign* of the agents' expected payoff differential under the original policy. Arguments similar to those establishing Theorem 1 in the main text then imply that, when the agents learn that their expected payoff differential

[31] A policy Γ is regular if MARP under Γ is well defined and the sign of the agents' expected payoff differential under MARP is measurable in θ.

[32] Because, in the generalized model, the default outcome need not be a deterministic function of θ, the definition of $x^*(\theta)$ and $\theta_0(x)$ in the main text must be amended as follows: $x^*(\theta)$ is the critical signal threshold such that, when agents pledge for $x > x^*(\theta)$ and do not pledge for $x < x^*(\theta)$, the agents' expected payoff differential $u(\tilde{\theta}, 1 - P(x^*(\theta)|\tilde{\theta}))$ changes sign at $\tilde{\theta} = \theta$; $\theta_0(x)$ is the critical fundamental threshold such that, when agents pledge of $\tilde{x} > x$ and do not pledge for $\tilde{x} < x$, the agents' expected payoff differential $u(\theta, 1-P(x|\theta))$ changes sign at $\theta = \theta_0(x)$. As in the baseline model, we assume that these functions are continuous.

under the original policy was positive, under the new policy, they all pledge, irrespective of their signals. Likewise, when they hear their payoff was negative, they all refrain from pledging. That the new policy makes the agents better off then follows from the fact that the agents' payoff differentials are non-decreasing in the size of the aggregate pledge. In the special case in which θ is a perfect predictor of the default outcome, because the sign of the agents' expected payoff differential is determined by the default outcome, perfect coordination is obtained by informing the agents of the default outcome, as in the baseline model. In this case, the ability to coordinate perfectly the market while inducing the same default outcome as under the original policy extends to an even richer class of economies. In particular, economies in which (i) agents' prior beliefs need not be consistent with a common prior, nor be generated by signals drawn independently across agents, conditionally on θ, (ii) the number of agents is arbitrary (in particular, finitely many agents), (iii) agents' have a level-K degree of sophistication, (iv) payoffs may be heterogenous across agents, and (v) the designer may disclose different information to different agents (see the document "Additional Material" on the authors' websites for details).

Next, consider part (b). As explained above, when $p(x|\theta)$ is log-supermodular and $u(\theta, 1-P(x|\theta))$ has the single-crossing property of Condition S4-FB, then, under MARP, the agents' strategies are monotone in their private signals, no matter the structure of the policy Γ and the shape of the induced common posterior, G. Arguments similar to those establishing Theorem 2 in the main text then imply that the new policy that perfectly coordinates the agents does not need to reveal anything more than the sign of the agents' expected payoff differential under the original policy.

Next, consider part (c). The pass/fail policy described above clearly makes all agents weakly better off. In general, it need not make the policy maker better off. However, when Condition S4-PC also holds, possible losses to the policy maker from inducing fewer agents to pledge in states in which the agents' expected payoff differential is negative are compensated by having more agents pledge in those states in which their expected payoff differential is positive. When this is the case, the new policy leads to a Pareto improvement.

Finally, consider part (d). As discussed in the main text, in general, the optimal pass/fail policy need not be monotone in θ. However, it is monotone when, in addition to the conditions in part (c), Condition M in the main text also holds. Q.E.D.