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UNIQUENESS OF NASH EQUILIBRIUM FOR LINEAR-CONVEX STOCHASTIC DIFFERENTIAL GAMES

by

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Abstract

The uniqueness of Nash equilibria is shown for a class of stochastic differential games where the dynamic constraints are linear in the control variables and the result is applied to an oligopoly.

I. INTRODUCTION*

Despite the many results available on nonzero-sum Nash differential games, we have very limited knowledge on the uniqueness of optimal strategies. In fact quite often uncountably many equilibria exist (Basar, 1977), although stochatic elements sometimes can shrink the set of equilibria considerably by rendering strategies which depend on the history of the game inoptimal (Basar and Olsder, 1982; Corollary 6.4). In a deterministic setting, one can alternatively require the strategies to be functions of time and the current state only. This approach is taken by Papavassilopoulos and Cruz (1979), who derive a uniqueness theorem for a class of analytic games of this type.

In the present paper we give a similar uniqueness result, which allows the state dynamics to be stochastic, but require them to be linear in the control variables. After deriving the result, we will apply it to a dynamic duopoly.

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II. A STOCHASTIC DIFFERENTIAL GAME

Notation

$i = 1,\dots,n$ generic player

$t \in [0,1]$, time

$s_t \in \mathbb{R}^m$, value of state variable at time t

U, a compact metric space.

A, space of measurable functions from $[0,1] \times \mathbb{R}^m$ to U

$P_i \in A$, (feedback) strategy of player i

$P \in A^n$, a set of strategies, one for each player

$P_{it} = P_i(t,s_t)$, value of control variables of player i for all (t,s_t)

$P_t' = (P_{it},\dots,P_{nt})$

B_1, space of C^2 functions from $[0,1] \times \mathbb{R}^m \times U^n$ to $\mathbb{R}^m$

B_2, subset of B_1 in which the functions are linear in the control variables, have bounded derivatives, and are bounded at some point, say $(t,0,0)$.

$F \in B_2$

$F(t,s_t,P_t)$, drift in s_t, for all (t,s_t,P_t)

z_t, Brownian motion in R^m

C, space of C^2 functions from $[0,1] \times \mathbb{R}^m$ to $\mathbb{R}^m$

$\sigma \in C$

$\sigma(t,s_t)$, $m \times m$ matrix, which has a bounded inverse, bounded derivatives, and is bounded at $(t,0)$

σ_{ij}, a typical element of $\sigma(t,s_t)$

D_1, space of C^2 functions from $[0,1] \times \mathbb{R}^m \times U$ to $[0,k_1]$, $k_1 \in \mathbb{R}_+$.

D_2, subset of D_1 in which the functions are convex in the control variables

$\pi_i \in D_2$

$\pi_i(t,s_t,P_{it})$ instantaneous, discounted payoff to player i for all (t,s_t,P_{it})

E, space of C^2 functions from $\mathbb{R}^m$ to $[0,k_2]$, $k_2 \in \mathbb{R}_+$

$\nu_i \in E$

$\nu_i(s_1)$, discounted terminal value of s_1 to player i

An N-Player Game

We can now define the game

$$\text{(G)} \quad \begin{aligned} &\min_{P_i} \int_0^1 \pi_i(t,s_t,P_{it})dt + \nu_i(s_1), \; i = 1,\ldots,n \\ &(1)\; ds_t = F(t,s_t,P_t)dt + \sigma(t,s_t)dz_t \\ &s_o \in \mathbb{R}^m \text{ given} \end{aligned}$$

Theorem 1: (G) has a Nash equilibrium.

Proof: This is a direct application of Corollary 1 in Uchida (1978). Q.E.D.

Noting that an equilibrium in general will depend on the initial condition, we can further get

Theorem 2: If (G) has a Nash equilibrium P^* in C^2 strategies, then that equilibrium is unique in that class of strategies.[1]

Proof: This complex proof consists of five steps.

1. Note first that the minimizing strategies are unique as functions of the arguments. That is, for any given $(t,s_t,\alpha_{it}) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^n$, the equations:

(2) $$\underset{P_{it}}{\text{Min}}\ \alpha_{it}^{'}F(t,s_t,P_t) + \pi_i(t,s_t,P_{it}),\ i = 1,\dots,n$$

have at most one solution P_t in U^n.

(To see this, remember that $F(\cdot)$ is linear in P_i such that (2) is additive in functions of the individual control variables, such that each player minimizes a convex function on R, independent of the actions of other players.)

2. By assumption, these strategies are C^2 and we can define value functions: V_i: $[0,1] \times \mathbb{R}^m \to \mathbb{R}$, $i = 1,\dots,n$, as

$$V_i(\tau,\phi) \equiv \underset{\tau,\phi}{E} \int_t^1 \pi_i(t,\tilde{s}_i(\tau,\phi,P^*),\ P_i(t,s(\tau,\phi,P_t^*)))\ dt + \nu_i(s_1(\tau,\phi,P^*))$$

where $\tilde{s}_t(\tau,\phi,P^*)$ is a realization of (1) starting from (τ,ϕ), with players using strategies P^*. Since all the functions defining $V_i(\cdot)$ are C^2 so is V_i.

3. By the Bellman principle, the V_i's therefore solve:

$$(3)\begin{cases} \dfrac{\partial V_i}{\partial t} + \dfrac{1}{2} \displaystyle\sum_{j,k=1}^{m} \sigma_{ij}^2 \dfrac{\partial^2 V_i}{\partial s_j \partial s_k} + \underset{P_i}{\text{Min}} \left[\sum_{j=1}^{m} \dfrac{\partial V_i}{\partial s_j} F_j(t,s_t,P_t) + \pi_i(t,s_t,P_{it})\right] = 0 \\ V_i(1,s) = \nu_i(s_1) \\ i = 1,\dots,n \end{cases}$$

4. (3) has a unique solution in the class C^2 (by Theorem IV. 10.1 in Ladyzenskaja et al., 1968).

5. So there can be only one V and this induces a unique P^*. Q.E.D.

III. EXAMPLE: A DIFFERENTIATED DUOPOLY

Model

A continuum of consumers are distributed evenly on [0,1]. Each has a demand curve of the form $\alpha - 2\beta\, \tilde{P}_i(s)$, $(\alpha,\beta) \in R_+^2$, where $\tilde{P}_i(s)$ is the "effective" price (price per unit quality) of firm $i = (0,1)$ to a consumer at $s \in [0,1]$. At any given time, all consumers in $[0,s_t]$ buy only from firm 0, whereas all consumers in $(s_t,1]$ buy only from firm 1. The marginal buyers flow, for them, to the most attractive firm, as:

$$\text{(ii)} \qquad ds_t = a(\tilde{P}_1(t,s_t) - \tilde{P}_0(t,s_t))dt - \sigma[s_t(1 - s_t)]^{1/2}dz_t$$

where $(a,\sigma) \in \mathbb{R}_+^2$ and z_t is Brownian motion.

The two firms are positioned at 0, firm 0, and firm 1, firm 1, and their effective prices are the products of the distance to the consumer and their nominal prices $(P_{0t},P_{1t}) \in \mathbb{R}^2$. So (ii) takes the form $ds_t = a[(1 - s_t)P_{1t} - s_t P_{0t}]dt + \sigma[s_t(1 - s_t)]^{1/2}dz_t$.

The sales of firm 0 are therefore given by $\int_0^{s_t}(\alpha - 2\beta P_{0t} r)dr = \alpha s_t - \beta P_{0t} s_t^2$, whereas the sales of firm 1 are $\alpha(1 - s_t) - \beta P_{1t}(1 - s_t)^2$. The firm's strategies are functions of (t,s_t).

We will solve the game over a unit time horizon, such that the firm's objectives are

$$\text{(4)} \qquad \max_{P_i} \int_0^1((\alpha s_{it} - \beta P_{it} s_{it}^2)P_{it})dt + 2s_{i1} - s_{i1}^2, \quad i = 0,1$$

where $s_{0t} = s_t$, $s_{1t} = 1 - s_t$ and $s_0 \in (0,1)$.

Results

The game (4), (ii) leads to value functions of the form

$V_i(s_{it}) = A(t)s_{it}^2 + B(t)s_{it} + C(t)$, $(i = 0,1)$, and the price functions:

$P_{it}^* = \frac{\alpha}{2\beta s_{it}} - \frac{aA(t)}{\beta} - \frac{aB(t)}{2\beta s_{it}}$, $i = 0,1$ where $A(t)$, $B(t)$ and $C(t)$ solves

$dA/dt + 3a^2A^2/\beta = 0$, $A(1) = -1$, $dB/dt + a^2AB/\beta - 2a^2A^2/\beta = 0$, $B(1) = 2$, and

$dC/dt + \sigma^2 A + (\alpha^2 - 4a^2AB - a^2B^2)/(4\beta) = 0$, $C(1) = 0$, respectively. From

this, $A(t) = -[1 + 3a^2(1 - t)/\beta]^{-1}$ and $P_{it}^* > 0$ for all t, s_t if α is

sufficiently large.

IV. CONCLUSION

In the present paper, the uniqueness of the feedback Nash strategies is shown for a class of stochastic differential games, where the state dynamics are linear in the control variables. Much, of course, needs to be done before the potential of this type of model can be realistically assessed. First, this is just uniqueness of Nash equilibria, although in very complex strategy spaces. If we allow more conjectural variations, the number of equilibria multiply again. Secondly, while the existence result of Uchida (1978) is reasonably general given the type of uncertainty he postulates, it needs extension to other types of uncertainties. Our own uniqueness result is much less general and should be relatively easy to extend. Finally there remains the problem of solving or characterizing solutions to the system (2), such that qualitative insights can be obtained.

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NOTES

1. Given the Markov property of the dynamic constraint it is obvious that no other equilibrium exist in the wider class of strategies which result if we allow P(•) to depend on the entire history of s.

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