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OPTIMAL CONSUMPTION OF A NONRENEWABLE RESOURCE WITH STOCHASTIC DISCOVERIES AND A RANDOM ENVIRONMENT

by

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Abstract

We present a general model for the optimal consumption of a nonrenewable resource under two kinds of uncertainties. One source of uncertainty is in the resource discovery process and the other is in the economic environment that affects resource supply and demand conditions, such as exhaustion and development of a substitute. The problem is formulated as one of optimally controlling a storage process with Markov additive discoveries. The optimal value of the resource stock is characterized as the solution of a functional equation and the existence of an optimal consumption policy is established. It is shown that, in a given environment, the optimal consumption rate is increasing and the resource price is decreasing in the level of proven reserves. A counterexample is provided to show that better environments may in fact mean higher prices and lower consumption rates. Finally, a variety of examples is given to illustrate the scope and applicability of the general model.

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1. INTRODUCTION

Since Hotelling's (1931) classic paper, there has been considerable research on economic decisions relating to nonrenewable resources; see, for example, the Review of Economic Studies (1974) symposium on the economics of exhaustible resources. However, only relatively recent studies have incorporated the important element of uncertainty into their models. In these models the uncertainty may be about (1) the total supply of the resource stock (e.g. Kemp (1976), Cropper (1976), Loury (1978) and Gilbert (1979)), (2) the process of discovering new supplies of the resource through exploration (e.g. Arrow and Chang (1980) and Deshmukh and Pliska (1980)), (3) the future demand for the resource (e.g. Weinstein and Zeckhauser (1975) and Pindyck (1980)), or (4) the timing of development of a producible substitute (e.g. Dasgupta and Heal (1974), Kamien and Schwartz (1978) and Dasgupta and Stiglitz (1981)). In Deshmukh and Pliska (1981) we have developed a single model that can be specialized to represent each of these uncertainties in the resource supply and demand conditions.

A shortcoming of all these models is their simplicity. By focussing on a single source of uncertainty, they are inherently less realistic than a model that would simultaneously capture all of the uncertain events of interest. However, the development and analysis of such a comprehensive model is a formidable undertaking, so as an intermediate step, we present in this paper a general model of nonrenewable resources that simultaneously captures two kinds of uncertainties. In particular, one source of uncertainty is in the resource discovery process and the other is in the economic environment that affects the resource supply and demand conditions outlined in the preceding paragraph. The problem is then to determine an optimal resource depletion policy in the presence of these multiple and possibly interrelated uncertainties.

We formulate this problem as a controlled storage process with Markov additive discoveries. Since our approach extends the one in Deshmukh and Pliska (1980), in Section 2 we briefly summarize the model and results from that paper. The present model that includes the environmental component is then explained in Section 3 and, utilizing theoretical results in Ozekici (1979), the properties of the optimal resource consumption rates and prices are derived in Section 4. It is shown that, in a given environment, the optimal consumption rate is increasing and the resource price is decreasing in the level of proven reserves. We also show, by example, a counterintuitive result that better environments may in fact mean higher prices and lower consumption rates. Section 5 considers certain special cases that illustrate the scope and applicability of our general model and Section 6 concludes the paper.

2. REVIEW OF THE PREVIOUS MODEL

In this section we briefly review our (1980) model, thereby establishing the background and notation for the rest of this paper. Let $X_t \geq 0$ denote the level of proven reserves of the resource at time $t \geq 0$. The resource is depleted according to a consumption rate policy $c: \mathbb{R}_+ \to [0, \bar{c}]$ and is augmented by stochastic discoveries that are controlled through an exploration rate (search intensity) policy $e: \mathbb{R}_+ \to [0,\bar{e}]$, where $\mathbb{R}_+ = [0, \infty)$ and $\bar{c}$ and $\bar{e}$ are finite upper bounds on the consumption and exploration rates. As a function of the resource level on hand at time t, the consumption policy specifies the consumption rate $c(X_t)$ and the exploration policy specifies the (search) effort rate $e(X_t)$. Naturally, $c(0) = 0$. If $e(X_t) = e$, discoveries of new stocks occur at a probabilistic rate $\lambda(e)$, i.e., $\lambda(e)dt$ is the approximate probability that a discovery will occur during $(t, t+dt)$. Given a discovery, its magnitude is determined by a probability measure $G(e, \cdot)$. Thus, if $e(X_t) = e$, $\lambda(e)\, G(e, [y, \infty))$ is the probabilistic rate at which new stocks of size at least y are discovered, and our basic assumption was that this quantity is nondecreasing in e for each $y \geq 0$, i.e., increased exploration effort expedites discoveries of larger stocks.

Let $D^0 = \{D_t^0, t \geq 0\}$ be the stochastic discovery process, where $D_t^0 \geq 0$ denotes the cumulative amount of the resource discovered by time t. Then the resource level process $X = \{X_t, t \geq 0\}$ is a Markov process controlled by policy $\pi = (c(\cdot), e(\cdot))$ and satisfying the storage equation:

$$(1) \qquad X_t = X_0 + D_t^0 - \int_0^t c(X_s)ds, \quad t \geq 0.$$

The resource consumption at rate c provides a social utility at rate $U(c)$, where the utility function $U: [0, \bar{c}] \to \mathbb{R}_+$ is assumed to be concave and nondecreasing, with $U(0) = 0$. The resource exploration effort rate e costs $h(e)$, where the exploration cost function $h: [0, \bar{e}] \to \mathbb{R}_+$ is continuous and nondecreasing with $h(0) = 0$.

With the discount rate $\rho > 0$, the total expected discounted net utility starting with the resource level $X_0 = x$ and following a policy π is given by

$$(2) \qquad v_\pi(x) = E\{\int_0^\infty e^{-\rho t}[U(c(X_t)) - h((e(X_t))]dt \mid X_0 = x\}.$$

The social planner's problem is then to choose from an admissible class of policies, A, an optimal $\pi^* = (c^*(\cdot), e^*(\cdot))$ that maximizes $v_\pi(\cdot)$ to yield the optimal value

$$(3) \qquad v(x) = \sup_{\pi \varepsilon A} v_\pi(x), \; x \geq 0.$$

Under suitable restrictions on A, it was shown in Deshmukh and Pliska (1980) that

(a) the optimal value function $v(\cdot)$ is the unique nonnegative, strictly increasing and concave solution of the dynamic programming optimality equation:

$$(4\text{-a}) \qquad \rho v(x) = \sup_{\substack{c\varepsilon[0,\underline{c}] \\ e\varepsilon[0,\bar{e}]}} \{u(c) - h(e) - cv'(x) + \lambda(e) \int_0^\infty [v(x + y) - v(x)]G(e,dy)\}, \; x > 0$$

and

$$(4\text{-b}) \qquad \rho v(0) = \sup_{e\varepsilon[0,\bar{e}]} \{- h(e) + \lambda(e) \int_0^\infty [v(y) - v(0)]\, G(e,dy)\},$$

(b) there exists an optimal policy $\pi^* \varepsilon A$, and under this policy the optimal consumption $c^*(\cdot)$ is nondecreasing and the optimal exploration effort $e^*(\cdot)$ is nonincreasing in the resource level, and

(c) the marginal contribution of the resource (i.e., the shadow price), $v'(X_t)$, is expected to rise at the social rate of discount ρ, which is a stochastic analog of Hotelling's (1931) fundamental result.

3. THE NEW MODEL

The purpose of this paper is to extend the previous model in order to include certain stochastic aspects of the economic environment which affect the resource supply and demand conditions. In the previous model, the process of discoverying new supplies (as summarized by the functions λ, G and h) was assumed to be stationary over time, thereby ruling out the possibilities of fewer and smaller discoveries (including total exhaustion) or cost savings due to advances in the exploration technology. Similarly, the utility function u (which also implicitly includes the extraction costs) was assumed to be independent of the environment, implying that the resource demand does not depend upon development of (partial or perfect) substitutes or advances in the extraction technology over time. We now wish to enrich this model by introducing a stochastic environmental component that affects the discovery process and the utility function and thus permits a consideration of the above-mentioned changes in the economic conditions. However, to keep the technical difficulties at a manageable level, the discovery process will be taken as being uncontrolled, so that the only decision variable in this paper will be the resource consumption rate.

Let Z_t denote the state of the economic environment at time t and suppose the environmental process $Z = \{Z_t, t \geq 0\}$ is a Markov process with a discrete state space E. If $Z_t = z$, the sojourn time in state z is exponentially distributed with mean $1/\mu(z)$, where μ is a real-valued nonnegative function on E satisfying $\sup_{z \varepsilon E} \mu(z) < \infty$. At the end of this random amount of time, the environmental state z changes to w with probability $Q(z, w)$, where the transition kernel (i.e., Markov matrix) $Q(\cdot, \cdot)$ satisfies $Q(z, z) = 0$ and $\sum_{w \varepsilon E} Q(z, w) = 1$ for all $z \varepsilon E$. If $\mu \equiv 0$, or if E is a singleton set, we obtain the constant environment model of the previous section as a special case.

The environmental state affects the discovery process as follows. Whenever the environment changes from z to w, an additional amount of the resource becomes available according to a probability distribution $F(z, w, \cdot)$, so that F is a transition kernel on $E \times E \times \mathcal{B}_+$, where $\mathcal{B}_+$ is the Borel σ - algebra on $\mathbb{R}_+$. Later on we shall provide several provocative examples illustrating the interplay between the state of the environment and the manner in which new supplies of the resource become available. For now, let D_t^1 represent the cumulative amount of the resource augmented as a result of the environmental changes by time t. Then $D^1 = \{D_t^1, t \geq 0\}$ is an increasing pure jump Markov process and will be called the <u>environmental change component</u> of the total discovery process.

Even when the environment does not change, new supplies of the resource may be discovered through exploration, as in the previous section. We shall assume throughout this paper that the exploration process cannot be controlled by the planner but may depend upon the state of the exogenous environment. Analogous to the notation of the previous section, suppose that if $Z_t = z$, new sources of supply of the resource are discovered at the probabilistic rate $\lambda(z)$ and that $G(z, \cdot)$ is the probability distribution of the magnitude of a discovery, if one occurs. Here λ is a nonnegative function on E satisfying $\sup_{z \in E} \lambda(z) < \infty$ and G is a transition kernel on $E \times \mathcal{B}_+$. Let D_t^2 denote the cumulative amount of the resource discovered in this manner by time t. Then the stochastic process $D^2 = \{D_t^2, t \geq 0\}$ is also an increasing pure jump process and will be called the <u>compound Poisson component</u> of the total resource discovery process. Note that D^2 is similar to D^0, the discovery process in the old model, except now it depends on the exogenous economic environment instead of the chosen exploration rate policy.

Finally, in addition to the above two stochastic discovery components D^1 and D^2, we shall also permit deterministic additions to the resource supply by the environment. Suppose that whenever $Z_t = z$, the resource level is continuously augmented at rate $d(z)$, where d is a nonnegative function on E satisfying $\sup_{z\varepsilon E} d(z) < \infty$. For technical reasons, we shall also assume that $\inf_{z\varepsilon B} d(z) > 0$, where $B = \{z\varepsilon E;\ d(z) > 0\}$. Note that this does not rule out the case of no deterministic inputs, i.e., $d \equiv 0$. Let $D_t^3 = \int_0^t d(Z_s)ds$ be the cumulative deterministic input by time t and call the continuous increasing stochastic process $D^3 = \{D_t^3,\ t \geq 0\}$ the <u>deterministic component</u> of the resource discovery process.

With $D_t = D_t^1 + D_t^2 + D_t^3$, the cumulative total amount of resource discovered by time t, (where $D_0 = 0$), the <u>discovery process</u> $D = \{D_t,\ t \geq 0\}$ is an increasing stochastic process as in the previous section, but it is non-Markovian, in general. The pair (Z, D) of the environment and discovery processes is an increasing Markov additive process in the sense of Çinlar (1972) and represents additions to the proven reserves of the resource over time. The resource depletion occurs at any time t due to consumption rate c_t that the planner selects as a function of the current environment Z_t and the resource level X_t. As in the previous section, let $c: E \times \mathbb{R}_+ \to [0, \bar{c}]$ denote the <u>consumption policy</u> that specifies the rate $c(z, x)$ at which the resource is depleted at time t if $Z_t = z\varepsilon E$ and $X_t = x \geq 0$. Under the policy $c(\cdot, \cdot)$, the resource level process $X = \{X_t,\ t \geq 0\}$ satisfies the storage equation:

(5) $$X_t = X_0 + D_t - \int_0^t c(Z_s, X_s)ds,\ t \geq 0,$$

which is similar to (1) in the previous section. Let $\underline{\underline{C}}$ denote the class of admissible consumption policies which will be specified more precisely in the next section. Then for any given $c \in \underline{\underline{C}}$, our model thus far is that of a storage process with Markov additive inputs, as in Çinlar (1973).

To consider the planner's problem of selecting an optimal consumption policy, suppose the resource consumption generates social utilities and costs which may also depend upon the economic environment and the resource level. Let $u: E \times [0, \bar{c}] \to \mathbb{R}$ denote the consumption <u>utility function</u>, so that $u(z, c)$ is the utility rate (net of any extraction costs) at time t if the environment is $Z_t = z \in E$ and the consumption rate is $c_t = c \in [0, \bar{c}]$. Similarly, let $r: E \times \mathbb{R}_+ \to \mathbb{R}$ denote the <u>reward function</u>, where $r(z,x)$ is the reward rate at time t if the environment is $Z_t = z\in E$ and the resource level is $X_t = x > 0$. Thus, at time t the net utility rate to the society is assumed to be given by $u(Z_t, c_t) + r(Z_t, X_t)$. Note that there are no sign restrictions on u or r, so one can also interpret these as costs; a later section will provide some examples. Also, additional restrictions will be placed on functions u and r in the next section.

Suppose future utilities and costs are discounted at rate $\rho > 0$, so that the total <u>expected discounted value</u> of starting with initial states $Z_0 = z\in E$ and $X_0 = x > 0$ and employing a consumption policy $c \in \underline{\underline{C}}$ is given by

$$(6) \qquad v_c(z,x) = E\{ \int_0^\infty e^{-\rho t}[u(Z_t, c(Z_t, X_t)) + r(Z_t, X_t)]dt \mid Z_0 = z, X_0 = x\}.$$

The planner's problem is then to choose a consumption policy $c \in \underline{\underline{C}}$ so as to maximize $v_c(\cdot, \cdot)$. Denote the <u>optimal value</u> as

$$(7) \qquad v(z, x) = \sup_{c\in\underline{\underline{C}}} v_c(z, x) \quad , z\in E,\ x > 0.$$

A consumption policy $c^* \varepsilon \underline{\underline{C}}$ is said to be an optimal policy if $v_{c^*}(z, x) = v(z, x)$ for all $z \varepsilon E$ and $x \geq 0$.

With the freedom of choosing the consumption policy $c \; \varepsilon \; \underline{\underline{C}}$, our nonrenewable resources model now becomes that of a controlled storage process with Markov additive inputs. Such a model has been studied by Ozekici (1979), and in the next section we shall draw extensively on his analysis.

4. THE MAIN RESULTS

Our objective is to characterize the optimal value function v, ensure that there exists an optimal consumption policy $c^* \; \varepsilon \; \underline{\underline{C}}$, and study its properties. To achieve this, we need to place restrictions on the class of admissible policies, $\underline{\underline{C}}$, and on the utility and reward rate functions, u and r.

A real-valued function c on $E \times \mathbb{R}_+$ is said to be an admissible consumption policy if it satisfies a number of conditions. First, c is nonnegative and bounded above by $\bar{c} < \infty$; we will also assume $\bar{c} > \sup_{z \varepsilon E} d(z)$, thereby permitting the consumption rate to exceed the deterministic input rate. Second, $c(z, 0) \leq d(z)$ for every $z \varepsilon E$, thereby ensuring that the resource level cannot be come negative. Third, for each $z \varepsilon E$, assume that $c(z, \cdot)$ is left continuous on the left-closed set $\{x \; \varepsilon \; \mathbb{R}\text{: } c(z, x) \geq d(z)\}$ and right continuous on the right-closed set $\{x \; \varepsilon \; \mathbb{R}\text{: } c(z, x) \leq d(z)\}$. Finally, assume that for each $z \varepsilon E$ and $x \; \varepsilon \; \mathbb{R}$ there exists some $t_1 > 0$ and some function f satisfying

$$f(t) = z + \int_0^t \left[d(z) - c\left[z, f(s)\right)\right]ds$$

for all $t \; \varepsilon \; [0, t_1)$. Let $\underline{\underline{C}}$ denote the set of all such admissible consumption policies.

The last two assumptions are made by Ozekici (1979) to ensure the existence of a unique Markov additive process corresponding to the control c. The third assumption is easy to check and apparently does not rule out any interesting controls. The fourth assumption is more subtle, but Ozekici (1979) showed that it is satisfied by c if (i) c satisfies the first three assumptions, (ii) for each $z \in E$, $c(z, \cdot)$ has a finite number of discontinuities on any finite interval, (iii) for each $z \in E$, the sets $\{x \in \mathbb{R}: c(z, x) > d(z)\}$ and $\{x \in \mathbb{R}: c(z, x) < d(z)\}$ are closed, and finally either (iv-a) for each $z \in E$, $c(z, \cdot)$ is nondecreasing or (iv-b) for each $z \in E$, $c(z, \cdot)$ is piecewise Lipschitz.

If $c(\cdot, \cdot)$ is an admissible policy and the resource level process X satisfies the storage equation (5) with $X_0 > 0$, then Ozekici (1979, Theorem II.4.2) has shown that (Z, X) is a Markov process with state space $E \times \mathbb{R}_+$. Thus, the choice of optimal $c^* \in \underline{\underline{C}}$ is a (continuous-time) Markov control problem.

As to the utility rate function $u: E \times [0, \bar{c}] \to \mathbb{R}$, assume that, for each $z \in E$, $u(z, \cdot)$ is continuous, concave and increasing. Also suppose that $u(z, \cdot)$ is twice continuously differentiable with the second derivative bounded away from zero. This latter assumption about the second derivative can be weakened somewhat, but then the results become more awkward and complicated (cf. Ozekici (1979, Assumption IV.1.3)). Similarly, regarding the reward rate function $r: E \times \mathbb{R}_+ \to \mathbb{R}$, assume that for every $z \in E$ the function $x \to r(z, x)$ is concave, increasing, bounded above and has a finite right-hand derivative at the origin. Our concavity and monotonicity assumptions on u and r are natural from the economic standpoint, while the others are necessary for technical reasons and expositional simplicity. With these assumptions, for any $c \in \underline{\underline{C}}$, the expected discounted value function v_c given by (6) is well-defined and bounded.

We are now ready to state our main results, which follow directly from a lengthy analysis in Ozekici (1979) culminating in his Corollary IV.1.4. This analysis is somewhat similar to that in Deshmukh and Pliska (1980) in that it involves first using dynamic programming methods and functional analysis to characterize the optimal value function v and its properties, then showing that there exists an optimal policy, and finally characterizing properties of this policy.

<u>Theorem A</u>: The optimal value function $v = \sup_{c \in \underline{\underline{C}}} v_c$ is the unique bounded differentiable solution of the optimality equation:

$$\text{(8-a)} \qquad \rho v(z, x) = \sup_{c \in [0,\bar{c}]} \left\{ u(z,c) + [d(z) - c] \frac{\partial v(z,x)}{\partial x} \right\}$$

$$+ r(z, x) + \lambda(z) \int_0^\infty [v(z,x + y) - v(z, x)] G(z, dy)$$

$$+ \mu(z) \sum_{w \in E} Q(z, w) \int_0^\infty [v(w, x + y) - v(z, x)] F(z, w, dy), \; z \in E, \; x > 0$$

and

$$\text{(8-b)} \qquad \rho v(z, 0) = \sup_{c \in [0, d(z) \wedge \bar{c}]} \left\{ u(z, c) + [d(z) - c] \frac{\partial v(z,0)}{\partial x} \right\}$$

$$+ r(z, 0) + \lambda(z) \int_0^\infty [v(z, y) - v(z, 0)] G(z, dy)$$

$$+ \mu(z) \sum_{w \in E} Q(z, w) \int_0^\infty [v(w, y) - v(z, 0)] F(z, w, dy), \; z \in E.$$

Furthermore, for every $z \varepsilon E$, the function $v(z, \cdot)$ is concave and increasing on $\mathbb{R}_+$. Finally, the function $g(\cdot) = \lim_{x \to \infty} v(\cdot, x)$ is the unique bounded function on E satisfying

$$(9) \qquad \rho g(z) = u(z, \bar{c}) + r(z, \infty) + \mu(z) \sum_{w \varepsilon E} Q(z, w) \left[g(w) - g(z)\right], \; z \varepsilon E.$$

The first term on the right hand side of the optimality equation (8) represents the effect of the continuous resource supply and consumption on the optimal value, the third term corresponds to the expected jump rate of change in the optimal value due to the compound Poisson component of the resource discovery process, and the last term accounts for the jump changes in the optimal value due to discoveries resulting from the environmental changes. If the environment is constant over time, then the last term vanishes and (without a deterministic input but with a controlled discovery process) we obtain the characterization of v in Deshmukh and Pliska (1980), summarized as result (a) and equation (4) in Section 2 of this paper. As in Deshmukh and Pliska (1980, 1981), $\frac{\partial v(z,x)}{\partial x}$ is the marginal contribution of an additional resource stock to total optimal value and is interpreted as the shadow price of the resource at the resource level x in the environment z. Since $v(z, \cdot)$ is concave, in any given environment $z \varepsilon E$, this price is decreasing in the resource level.

Our next result is similar to the one summarized in (b) of Section 2.

<u>THEOREM B</u>: There exists an optimal consumption policy $c^* \varepsilon \underline{\underline{C}}$ yielding $v_{c^*} \equiv v$ and it specifies in state (z, x) a consumption rate $c^*(z, x)$ that attains the supremum on the right hand side of the optimality equation (8). For every $z \varepsilon E$, the function $c^*(z, \cdot)$ is nondecreasing on $\mathbb{R}_+$.

The first statement follows from Ozekici (1979) and then the second statement follows from (8) together with the concavity of $u(z, \cdot)$ and $v(z, \cdot)$. Thus, in any environment, the greater the resource stock on hand the faster it should be consumed. Equivalently, since the shadow price $\frac{\partial v(z,x)}{\partial x}$ is nonincreasing in the resource level x, the lower the shadow price of the resource the higher is the optimal consumption rate. Also note that the characterization of c* from the optimality equation (8) yields $\frac{\partial u\left(z, c^*(z,x)\right)}{\partial c} = \frac{\partial v(z, x)}{\partial x}$, i.e., at the optimum, the marginal utility of consumption equals the marginal value of the resource stock.

A next natural problem is to investigate the dependence of the shadow price $\frac{\partial v(z, x)}{\partial x}$, on the economic environment z, which will enable us to determine how the optimal consumption rate $c^*(z, x)$ depends upon the environment z. The actual relationship will depend upon a particular interpretation of the environment and the stochastic process governing it. In general, we would like to compare environments and ask whether, for a given resource level, the optimal consumption rate is higher and the shadow price is lower in "better" environments. Unfortunately, as the remainder of this section shows, a general comparison of different environments is not easy, and even in specific cases the results may turn out to be highly counterintuitive.

The main difficulty in comparing different states of the environment is that each can be characterized on the basis of more than one criterion. Associated with each environmental state z, for example, are two return functions $\left(u(z, \cdot) \text{ and } r(z, \cdot)\right)$ as well as three separate components $\left(\mu(z)F(z, \cdot, \cdot), \lambda(z)G(z, \cdot) \text{ and } d(z)\right)$ of the discovery process. Therefore, even though one state might be clearly superior to another one with respect to one criterion, the reverse might be true with respect to another criterion, in which case there is no obvious way to decide which state is better. A further difficulty in comparing states is due to the fact that the environment changes from one state to another (according to $\mu(\cdot)$ and $Q(\cdot, \cdot)$). For example, suppose that a state z is clearly superior to a state w according to all of the above-mentioned criteria but a third and substantially superior state y is accessible from w but not from z. Then it may be preferable from a long-run viewpoint to be in state w than in state z.

One approach to circumventing these difficulties would be to compare two environmental staes that differ on the basis of only one criterion but are identical in all other respects (including the probability laws governing the environmental changes from those states). For example, if $d(z) > d(w)$, all other things being the same in the two states, then we could say that state z is better than state w (in the long-run as well in the short-run). Another, and more general, approach would be to identify a single criterion that properly summarizes the competing effects of all factors governing the utilities and probability laws. The leading candidate for such a criterion is the maximum expected discounted total value function v. This criterion cannot always be applied because v is not a scalar; for fixed z it is a function on $\mathbb{R}_+$. However, it does seem reasonable to assert that the environmental state z is better than state w if $v(z, x) \geq v(w, x)$ for all $x \geq 0$, i.e., starting in the environmental state z provides a higher long-run optimal return than starting in w, for all initial resource levels.

With such a reasonable single criterion for comparing environments, we would then like to show, for example, that in better environments the optimal consumption rates are higher and the shadow prices are lower. However, the following counter-example shows that this result may not hold in general. First, suppose $E = \{z\}$ (a singleton), $\lambda(z) = \mu(z) = 0$ and $d(z) = 1/4$, so that we have only deterministic resource inputs and a static environment. Let the utility and reward rate functions be given by

$$u(z, c) = c^{1/2} \text{ and } r(z, x) = \begin{cases} kx & , \quad 0 \leq x \leq 1 \\ k & , \quad x \geq 1. \end{cases}$$

where the parameter $k > 0$. With $\rho = 1$ let

$$\bar{x} = \inf \left\{x \geq 0; \frac{\partial r(z, x)}{\partial x} < \rho \frac{\partial u(z,dz)}{\partial c} \right\} = \begin{cases} 0, \text{ if } k < 1 \\ 1, \text{ if } k \geq 1 \end{cases} .$$

According to Ozekici (1979, Theorem II.3.45), the optimal consumption rate satisfies $c^*(z, x) \leq d(z) = 1/4$ for all $x \leq \bar{x}$ and $c^*(z, x) \geq d(z) = 1/4$ for all $x \geq \bar{x}$.

Now suppose $k \geq 1$, so that $\bar{x} = 1$. Then we must have $c^*(z, x) < 1/4$ for some $x < 1$. To see this, suppose instead that $c^*(z,x) = 1/4$ for all $x \leq 1$. Then $X_t = x$ for all $t \geq 0$, whenever $X_0 = x \leq 1$, in which case

$$u\big(z, d(z)\big) + r(z, x) = kx + 1/2 \quad , \ x \leq 1$$

is the reward rate. Furthermore, since $\rho = 1$,

$$v_{c^*}(z, x) = kx + 1/2 \quad , \quad x \leq 1.$$

If this policy is optimal, then v_{c^*} must satisfy the functional equation (8), which specializes to

$$\rho v_{c^*}(z, x) = \sup_{c \in \underline{\underline{C}}} \left\{u(z, c) + \left[d(z) - c\right] \frac{\partial v_{c^*}(z, x)}{\partial x} \right\} + r(z, x).$$

Substituting and differentiating, one computes the maximizing value of $c^*(z, x) = 1/(4k^2)$. Substituting this back into the right hand side of the functional equation, one sees that it is satisfied if $k=1$ but it is false for all $k > 1$. Thus $c^*(z, x) = d(z) = 1/4$ cannot be the optimal consumption rate for all $x \leq 1$, if $k > 1$, i.e., we must have $c^*(z, x) < 1/4$ for some $x < 1$. This preliminary example shows that it may be optimal to let the resource level rise by forgoing consumption over the short-run, in order to benefit from more favorable values of the rewrd rate function r over the long run.

Now consider a two-state model with $E = \{z, w\}$ and $\mu(z) = \mu(w) = 0$, so that states z and w do not communicate. The rest of the data is the same as in the preceding deterministic case, except now the parameter k (which depends upon the environmental state) is greater than 1 for state z and less than 1 for state w. Consequently, we have $r(z, x) \geq r(w, x)$ and $v(z, x) \geq v(w, x)$, so that state z is better than state w in terms of both the short-run as well as the long-run considerations. However, by the analysis in the preceding deterministic case, the optimal consumption rate satisfies $c^*(z, x) < d(z) = 1/4$ for some $x < 1$, whereas $c^*(w, x) \geq d(w) = 1/4$ for that same x (the $\bar{x}$ for w equals zero). Thus, the optimal consumption rate is actually lower in the better environmental state. Furthermore, since from the optimality equation we have $\frac{\partial v(\cdot, x)}{\partial x} = \frac{\partial u(\cdot, c^*(\cdot, x))}{\partial c}$, concavity of u implies that the shadow price at some level x will be higher in state z than in state w. What seemed to be the better state turns out to have higher prices and lower consumption rates.

Thus, in certain cases, the dependence of prices and consumption rates on the economic environment may turn out to be counterintuitive. This is why it is very difficult, if not impossible, to obtain general results comparing different states of the environment. Nevertheless, for some other specific cases, such as those considered in the next section, it may be possible to obtain these kinds of results.

5. ILLUSTRATIVE EXAMPLES

We now specialize the general model of Section 3 to illustrate how it can capture a variety of economically meaningful situations by appropriately interpreting the environmental component. In the first four examples the environment affects the resource supply and in the last example it also affects the resource demand conditions. In each case, Theorems A and B of Section 4 hold, so that the optimal value function is characterized by an appropriate specialization of the optimality equation, there exists an optimal consumption policy and, for any given environmental state, the resource consumption rate is nondecreasing and the resource price is nonincreasing in the level of proven reserves. We also indicate how the resource consumption/price may change with the environment. However, we do not provide any detailed analyses or proofs, as they appear to pose difficult and lengthy digressions.

5.1 An OPEC Model

Suppose the economy must depend solely upon an external supplier who controls the continuous resource supply rate, $d(\cdot)$, and/or the resource price denoted as $p(\cdot)$. If $Z_t = z$, he supplies at a constant rate $d(z) \geq 0$ and charges price $p(z) \geq 0$, where $z \in E = \{0, 1, ...\}$. Suppose higher values of z correspond to worse environments in terms of lower supplier rates and/or higher prices. The supplier's state of willingness to supply changes according to a Markov process with data $\mu(\cdot)$ and $Q(\cdot, \cdot)$. There are no other sudden discoveries of the resource, so that $\lambda \equiv 0$ and $F(z, w, \cdot)$ is concentrated at 0. If $u_1(\cdot)$is the consumption utility function, then we have $u(z, c) = u_1(c) - p(z)$ and $r \equiv 0$. In this case, one expects lower consumption rates (higher shadow prices) in worse environments, i.e., $c^*(z + 1, x) \leq c^*(z, x)$ and $\frac{\partial v(z + 1, x)}{\partial x} \geq \frac{\partial v(z, x)}{\partial x}$.

5.2 A Nationalization Model

Suppose the resource stock lies in a foreign ground and becomes available to the economy through stochastic discoveries over time but (in the spirit of Long(1975)) may be suddenly expropriated due to nationalization at a random time, thereby abruptly terminating the discovery process. Let $E = \{0, 1\}$ and suppose environmental state $Z_t = 0$ denotes that the discovery process is "on" at time t and $Z_t = 1$ represents the event of nationalization by t. The random time, T, of nationalization is exponetially distributed with mean $1/\mu(0)$ and the state of nationalization is a trapping state, i.e., $\mu(1) = 0$. Upon nationalization, there are no new discoveries (i.e., $\lambda(1) = 0$) and the planner's problem is then the deterministic (Hotelling's) problem of optimally consuming the stock X_T over $[T, \infty)$. Before nationalization, discoveries occur according to a compound Poisson process with data $\lambda(0)$ and $G(0, \cdot)$, and there are no other resource inflows, so that $d \equiv 0$ and $F(z, w, \cdot)$ is concentrated at 0. The utility rates are $U(\cdot, c) \equiv u_1(c)$ and $r \equiv 0$. Again one expects $c^*(0, x) > c^*(1, x)$, so that the consumption rate is higher prior to nationalization.

5.3 An Exhaustion Model

In the preceding model, the discovery process was terminated exogenously and abruptly. Suppose instead that the discovery process slows down and eventually terminates because the finite but uncertain amount of the total resource stock in ground gets depleted and exhausted over time. The integer valued random variable S denotes the total stock size and $P(\cdot)$ is the planner's subjective probability mass function of S at time $t = 0$. Let the "environment" Z_t denote the cumulative amount of the resource discovered by time t. Then the termination date is a random variable (stopping time) $T = \inf \{t > 0;\ Z_t = S\}$, and again on $[T,\infty)$ we have the Hotelling problem of optimally consuming X_T without the possibility of further discoveries.

By the definition of the environment, the times and amounts of increases in the environment coincide with those in the resource level so that $D_t^1 = Z_t$. To describe the environmental process in detail, note that if $Z_t = z$ and $S = s > z$, then $(s-z)$ is the amount of resource yet to be discovered, a quantity that, in general, affects the rate and magnitude of new discoveries. To describe exploration technology, suppose $\nu(s-z)$ is the probabilistic rate of discoveries and $H(\cdot|s-z)$ the probability distribution of the size of a discovery. Here $\nu(0) = 0$ and $H(\cdot|s-z)$ is concentrated on $\{0,1,\ldots s-z\}$. A reasonable assumption is that for any u, $\nu(w) \sum_{y=u}^{\infty} H(y|w)$ is nondecreasing in w, i.e., the greater the stock remaining to be discovered, the faster are the discoveries of larger stocks. Given $Z_t = z$, the planner knows S is greater than z and so revises the probability distribution of S to $P(\cdot)/\sum_{w=z}^{\infty} P(w)$ on $\{z,z+1,\ldots\}$. This yields the expected probabilistic rate of discoveries, namely

$$\mu(z) = \sum_{s=z}^{\infty} \nu(s-z)\ P(s)/\sum_{s=z}^{\infty} P(s)$$

Similarly, the probability that the next discovery is of size y, i.e., that the environment changes from z to (z+y), is

$$Q(z,z+y) = \sum_{s=z+y}^{\infty} H(y|s-z)P(s)/\sum_{s=z}^{\infty} P(s).$$

Finally, since the amount of change in the environment is the same as the quantity of resource discovered, we have $F(z,z+y,\cdot)$ concentrated at y.

With these specifications of μ, Q and F, describing the environmental process, we have $D_t^1 = Z_t$. There are no other inputs, so $d \equiv \lambda \equiv 0$ and $D_t^2 = D_t^3 = 0$ for all t. Under the stated assumption on $\mu(\cdot)Q(\cdot,\cdot)$ one would expect to become more conservative as more of the resource is exploited, i.e., $c^*(z+1,x) < c^*(z,x)$.

This model of uncertain exhaustion extends the ones in the literature (e.g. Kemp (1976)) cited in Section 1 in that in addition to the uncertainty about the total stock size, we have also included the uncertainty regarding the discovery process through which the stock becomes available. The model also extends our previous one in Deshmukh and Pliska (1980) in that it now permits slower and smaller discoveries and eventual exhaustion of the resource stock over time.

5.4 An Exploration Model

As a possible scenerio for the discovery process, suppose the total land area is divided into discrete test sites and more promising sites are explored first. Let $Z_t \in E=\{0,1,2,...\}$ denote the number of test sites explored by time t, so that $Q(z,z+1) = 1$. If $Z_t = z$, let $1/\mu(z)$ be the mean time required to complete the exploration of the site (z+1) and let $F(z, z+1, \cdot)$ denote the probability distribution of the resource stock discovered at the site (z + 1); there are no other inputs, so that $\lambda \equiv d \equiv 0$. A reasonable assumption is that more promising sites can on average be explored faster and yield larger supplies, i.e., $\mu(z)$ is decreasing in z and $F(z,z+1,\cdot)$ is stochastically decreasing in z. Equivalently, as in Arrow and Chang (1980), the total land area may be finite and, as more area is explored, less remains available for further exploitation and hence less fruitful the exploration process becomes. In any case, we expect greater conservation of the resource as exploration proceeds over time, i.e., $c^*(z+1,x) < c^*(z,x)$.

5.5 A Model with Technological Advances

Let Z_t denote the technological state of the economy that affects the resource demand or supply conditions at time t. For example, Z_t may represent the state of the exploration or extraction technology and higher states may correspond to better technologies in terms of lower costs of exploration or extraction, i.e., $r(z+1,x) > r(z,x)$ or $u(z+1,x) > u(z,x)$. Alternatively, as in Dasgupta and Heal (1974), Z_t may represent the availability of a producible substitute (e.g., an electric car) for the resource, and suppose that higher values of Z_t correspond to availabilities of closer substitutes, yielding a lower marginal utility of consumption, i.e., $\frac{\partial u(z+1,c)}{\partial c} < \frac{\partial u(z,c)}{\partial c}$.

In either case, the environment changes due to technological advances that involve time delays and R&D expenditures. If $Z_t = z$, the technology changes after an exponential length of time with mean $1/\mu(z)$, requires the interim R&D expenditures at rate $[-r(z,x)]$, and yields an improved technology $(z+y)$ with probability $Q(z,z+y)$. The only resource availabilities are due to stochastic discoveries that occur according to the compound Poisson component with λ and G that are independent of Z, and we have $d \equiv 0$ and $F(z,z+y,\cdot)$ concentrated at 0. Then we expect faster resource consumption and lower prices in better technological environments, i.e.,
$c^*(z+1,x) > c^*(z,x)$ and $\frac{\partial v(z+1,x)}{\partial x} < \frac{\partial v(z,x)}{\partial x}$.

6. Concluding Remarks

We have presented a general model of nonrenewable resource consumption in the presence of two interrelated sources of uncertainty. First, the exploration process of discovering new supplies involves uncertainty and second, the economic environment that affects the discovery process and/or the consumption utility/cost function may be stochastic. It is shown that the maximum total expected discounted net utility is characterized as the unique solution of the dynamic programming optimality equation, that there exists an optimal consumption policy and that, under this policy, higher levels of the resource imply lower prices and higher consumption rates. We have also illustrated the model by indicating how it can be specialized to capture, among others, the possibilities of the resource exhaustibility or a substitute availability.

Unfortunately, it seems difficult to show how the optimal value function, the shadow price and the optimal consumption rate depend on the environmental state in economically meaningful ways. We look for future research to analyze these issues for specific cases, such as the ones we mentioned above.

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