The Ordinal Nash Social Welfare Function

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A social welfare function entitled 'ordinal Nash' is proposed based on risk preferences and assuming a common, worst social state for all individuals. The crucial axiom in the characterisation of the function is a weak version of IIA, in which only relative risk positions with respect to the worst state are considered. Thus the resulting social preference takes into account non EU risk preference intensity by directly comparing certainty equivalent probabilities. The function provides an interesting interpretation of the Nash-like utility product principle in the realm of social choice. However, necessary and sufficient conditions over the function domain state that all individuals should have the same probability distortion functions in their preference representations (clearly satisfied in the EU case).

Key Words: Nash program, social choice, risk preferences, non expected utility.

1. INTRODUCTION

In his classic work, Arrow [2] introduced the concept of a social welfare function. Arrow's axiomatic analysis led to the well known impossibility result, according to which no social welfare function exists that satisfies the axioms of universal domain, Pareto principle, independence of irrelevant alternatives (IIA) and nondictatorship. Arrow's pathbreaking conclusion was extensively analyzed thereafter, resulting in a variety of modified impossibility and possibility theorems. An interesting question concerns the axiomatization of the Nash social welfare function - the social aggregation

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rule which assumes the existence of a common worst social state for all individuals in society, and ranks social states according to the product of individual utility differences with respect to the common worst social state.

This paper reexamines the axiomatic justification for the Nash social welfare function. The interest in this question emerges from the need to clarify the role of the common worst social state, or origin, in the social aggregation rule, as opposed to its other two main properties. The first property is the individual cardinal utility assumption, meaning that individual utilities represent preferences uniquely up to a positive affine transformation. The second property is the non-comparability of cardinal utilities, which means that social aggregation is equal for cardinally equivalent individual utility profiles. Defining cardinally equivalent profiles such that the affine transformations are not necessarily interdependent across individuals, such a property is referred to as an assumption of no interpersonal comparability of cardinal individual welfare. It is well known, as shown by Sen [13], that without the origin assumption there is an impossibility result for cardinality and non-comparability. This impossibility result is analogue to Arrow's impossibility, which results from ordinality and non-comparability.

One way to resolve the individual cardinal utilities impossibility, that was discussed extensively in the social choice literature, is to weaken the non-comparability assumption to unit comparability or full comparability. These assumptions strengthen the cardinally equivalent utility profiles requirement, by imposing a connection between the positive affine transformations of individual utilities, such as all having the same positive multiplier in order to attain unit comparability and further requiring that all possess the same added constant for full comparability. Adding an anonymity axiom, these weaker assumptions were shown to characterize the utilitarian (d'Aspremont & Gevers, [3]) and leximin (Deschamps & Gevers, [4]) aggregation rules. A utilitarian rule is represented by the sum of the individual utilities, while leximin is similar to maxmin in the sense that both rules prefer states that are better for the worst off individuals and therefore both imply egalitarianism (leximin adds to maxmin in the case of ties using a lexicographic ranking[1]).

It was pointed out by Sen (in the book previously cited [13]), that a different escape route from cardinality with noncomparability could be found within a framework similar to Nash's bargaining theory [10] through the restriction of the domain to profiles with an origin. This method of achieving a possibility result shows the extent to which the origin assumption is crucial for the Nash social welfare function. Indeed, one can easily check that

[1]For example, maxmin declares the state valuation (5,1,8) indifferent to (1,4,7) since in both states the worst off individual have utility level 1, which considering anonymity violates the Pareto principle. To the contrary, leximin ignores the worst off individual in the case of ties and ranks according to the second worst off individual, and so on.

non-comparability of cardinal utility still holds for the Nash social welfare function, since utility origins are cancelled out, while the function's multiplicative form renders the utility units irrelevant. It would appear that the possibility result draws not on cardinal but ordinal comparability that utilizes the origin. Moreover, it is a modest type of ordinal comparability since it doesn't imply egalitarianism, as does full ordinal comparability. A natural question arises regarding the implications of assuming this kind of ordinal origin comparability without the cardinality assumption. One acceptable interpretation of cardinal utilities is vNM utilities that represent expected utility risk preferences. The current axiomatization of the Nash social welfare function, due to Kaneko & Nakamura [8], relies heavily on the expected utility assumption.

The first aim of this paper is therefore the analysis of axioms that characterize the Nash social welfare function based on risk preferences, while questioning the necessity of the expected utility assumption that implies cardinality. Apart from assuming the existence of a common, worst social state, the crucial axiom in the characterization of the function is a weak version of IIA, in which only relative risk positions with respect to the worst state are considered. The second aim is to provide an interpretation for the social aggregation rule that emerges from each preference profile separately but without abandoning the multi-profile approach. The current justification for the Nash social welfare function is given by axioms that characterize it, some of which are inter-profile axioms. Nevertheless, without looking at the axioms but considering each profile separately, it is not clear why social states should necessarily be compared using a utility difference multiplication rule. Exploiting the ordinal approach in the construction of the social preference, it is redefined without the use of utility multiplication, but instead in a way that individuals can rationalize directly in terms of the given preference profile. Consequently, the social preference is built directly from individual ordinal information.

The results include an axiomatization of a social welfare function entitled '*ordinal Nash*', which is defined in terms of individual preferences and alternatives, without the use of utilities, by comparing certainty equivalent probabilities. This function coincides with the Nash social welfare function for the case of expected utility preferences and thus provides an interesting alternative definition for the Nash social aggregation rule. Furthermore, it enhances its applicability by extending the domain to cases in which individuals exhibit non-EU risk preferences. This evaluation is undertaken directly when the social states being compared are different for precisely two individuals. When a greater number of individuals show interest in the comparison, it is undertaken indirectly following a sequence of intermediate steps, in which precisely two individuals are involved at each step.

The domain of preference profiles being investigated includes expected utility preferences, as well as more general preferences that have a multiplicatively separable representation for lotteries over two social states. The analysis utilizes the notion of induced utilities, as introduced in [7]. The results include conditions on the domain of preference profiles, which ensure that the ordinal Nash social welfare function is well defined, i.e. it yields a complete and transitive binary relation over the set of social states. For a society of size two, we define necessary and sufficient conditions that characterize the domain of the function, specifically as a result of the transitivity requirement. These conditions state that the probability distortion functions of the preference representations should be equal for all individuals. For a larger society, we define similar necessary and sufficient conditions for the transitivity requirement and sufficient conditions for the completeness property. The latter conditions include smoothness of the representation functions and a mild convexity assumption.

It is interesting to note that this result is rather similar to the general conclusion reached within the social choice literature discussing the aggregation of individual preferences under subjective expected utility (SEU) theory. This line of inquiry aims at aggregating both outcome utility functions and subjective probabilities. When SEU axioms are imposed on both social and individual preferences, it is shown that the strict Pareto principle does not generally hold. Mongin [9] shows within the framework of Anscomb-Aumann SEU theory [1] that under sufficient conditions of individual preference diversity (when the outcome utility functions are affinely independent), the axioms of SEU and strict Pareto are consistent only when all individual subjective probability beliefs are identical. This result and other similar work may be compared with our conclusion that the social aggregation requirements are consistent only when all individual preferences are compatible with the same objective probability distortion function over the set of elementary lotteries.

The analysis was motivated by the reinterpretation of the classic Nash bargaining solution, which was introduced by Rubinstein, Safra and Thomson (RST [12]) as the '*ordinal Nash*' bargaining solution. This solution was defined directly in terms of preferences and physical alternatives and was characterized by an outcome that is immune to all possible appeals. Moreover, the solution was extended to a domain of problems for which individual preferences did not necessarily satisfy EU assumptions.

The paper is organized as follows: section 2 states the social welfare axioms, section 3 characterizes the social welfare function for the case of two individuals, section 4 specifies the conditions for a general, finite size of society and section 5 develops an axiomatization of the function over the domains being investigated.

2. SOCIAL WELFARE AXIOMS

Consider a finite society of individuals, denoted by $\mathbf{N} = \{1, ..., n\}$, associated with a fixed, non-empty set $\mathbf{X}$ of possible social states and a social state $\mathbf{x}^0$ (origin), which is considered the worst for all individuals in the society. Each member k of the society has a preference relation $\succsim_k$ over $\mathcal{L}$, the set of simple (finite) lotteries over $\mathbf{X} \cup \{\mathbf{x}^0\}$. A lottery in $\mathcal{L}$ is denoted by $(\tilde{\mathbf{x}}, \mathbf{p}) = (\mathbf{x}^1, \ldots, \mathbf{x}^m; p^1, \ldots, p^m)$, $\sum_{k=1}^m p^k \leq 1$, with the convention that p^k is the probability of $\mathbf{x}^k \in \mathbf{X}$ and $1 - \sum_{k=1}^m p^k$ is the probability of $\mathbf{x}^0$. When $m = 1$, the lottery is called an *elementary* lottery and is denoted $p\mathbf{x}$. The subset of elementary lotteries is denoted by $\mathcal{E}$. As usual, $\sim_k$ and $\succ_k$ denote the symmetric and asymmetric components of $\succsim_k$, respectively.

The social welfare function we consider maps from a domain of risk preference profiles to a range of social preferences over $\mathbf{X}$. Thus we are not concerned with social risk preferences. Instead, individual risk preferences are used as an information basis for social ranking. Furthermore, the axioms we consider for the social welfare function have the strongest implications when the domain includes profiles that satisfy comprehensiveness with respect to $\mathbf{x}^0$. In other words for such profiles, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ if $\mathbf{x} \succsim_i \mathbf{y}$, then for some state $\tilde{\mathbf{y}} \in \mathbf{X}$ the preference profile satisfies $\mathbf{y} \sim_i \tilde{\mathbf{y}}$ and $\mathbf{x} \sim_k \tilde{\mathbf{y}}$ for each $k \in \mathbf{N} \backslash \{i\}$. For example, profiles that satisfy this assumption can be viewed as exhibiting free disposal of individual welfare. In order to present the results clearly and simply, attention will be restricted to such domains by assuming a specific structure for the set $\mathbf{X}$. More specifically, we assume that $\mathbf{X} \cup \{\mathbf{x}^0\} \subseteq \mathbb{R}_+^n$ is a compact and $\mathbf{x}^0$-comprehensive set, i.e. $\mathbf{x}^0 \in \mathbf{X}$ and for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$, $\mathbf{x}^0 \leq \mathbf{y} \leq \mathbf{x} \in \mathbf{X}$ implies[2] $\mathbf{y} \in \mathbf{X}$. Furthermore, (1) for every $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \geq \mathbf{x}^0$, (2) there exists $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} > \mathbf{x}^0$ and (3) $x_i^0 = x_j^0$ for every $i, j \in \mathbf{N}$. We also assume that for every $\mathbf{x} \in \mathbf{X}$ the preference relation $\succsim_k$ depends only on the kth coordinate x_k. A lottery $\ell \in \mathcal{L}$ can therefore be described by its marginal distributions n-tuple $(\ell_1, ..., \ell_n)$. Let $l = \max_k(\max\{x_k \mid \mathbf{x} \in \mathbf{X}\})$. Without loss of generality, preference relation $\succsim_k$ is considered to be defined on $\mathcal{L}([x_k^0, l])$, the set of lotteries over $[x_k^0, l]$. Let $\mathcal{P}$ denote the set of all preference relations that satisfy this condition and are complete, transitive, monotone with respect to the relation of first-order stochastic-dominance (with strict monotonicity when elementary lotteries are considered) and continuous (with respect to the topology of weak convergence) on the set of elementary lotteries.

We start by stating the axioms that characterize a social welfare function $W(\cdot)$. The first axiom concerns the function domain. Let $\mathcal{P}^{ms} \subseteq \mathcal{P}$ be the set of preference relations that have multiplicatively separable representations over elementary lotteries, that is, the value of an elementary lottery

[2]We use the following notations: $\mathbf{x} \geq \mathbf{y} \Leftrightarrow x_i \geq y_i, \forall i \in \mathbf{N}$ and $\mathbf{x} > \mathbf{y} \Leftrightarrow x_i > y_i, \forall i \in \mathbf{N}$.

px is given by $g(p)v(x)$, where g, v are strictly monotonic, $g : [0,1] \to [0,1]$ is onto and $v(\mathbf{x}^0) = 0$. The preference set $\mathcal{P}^{ms}$ contains the set of EU preference relations, as well as the set of '*disagreement linear*' (DL) preferences introduced by Grant and Kajii [5] (origin linear in the social choice framework). DL preferences have a linear representation over the set of elementary lotteries (in both cases of EU and DL preferences, $g_i(p) = p$, $\forall p \in [0,1]$). Furthermore, the set $\mathcal{P}^{ms}$ includes the entire family of '*rank-dependent utility*' (RDU) preferences (see Quiggin [11] and Weymark [14]) and Gul's [6] '*disappointment aversion*' (DA) family (see specific definitions in [7]). The function domain is assumed to be a subset of the n-cartesian product of the set $\mathcal{P}^{ms}$ that includes individual preference profile permutations. Let π denote a permutation of the individuals $(1, ..., n)$.

Definition 2.1. **MS** (Multiplicatively Separable Risk Preferences): W is defined over a non-empty domain $\mathcal{D} \subseteq (\mathcal{P}^{ms})^n$ such that for any permutation π of only two individuals, there exist $\langle \succsim_k \rangle_{k=1}^n, \langle \succsim_{\pi_k} \rangle_{k=1}^n \in \mathcal{D}$.

The second and third axioms are the standard weak order assumption and strict Pareto optimality condition.

Definition 2.2. **WO** (Weak Order): For each $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ is a complete and transitive binary relation over $\mathbf{X}$.

Definition 2.3. **PAR** (*Pareto Optimality*): *Let* $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ *and* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, *such that* $\mathbf{x} \succsim_k \mathbf{y}$ *for each* k *and* $\mathbf{x} \succ_j \mathbf{y}$ *for some* j. *Then,* $\mathbf{x} \succ \mathbf{y}$.

The forth axiom is the standard anonymity condition, in which the social preference is independent of the individuals' name or order. Thus exchanging individual preferences and the corresponding coordinates in the social state vectors should not change the social preference. Note that this axiom relates permuted vectors in $\mathbf{X}$ with permuted preference profiles, which is a result of the assumption that individual preferences depend only on one coordinate of each vector. This justifies the assumption stated above that the origin must be a vector with equal components. For any permutation π of the individuals $(1, ..., n)$ and any $\mathbf{x} \in \mathbf{X}$, denote $\langle x_{\pi_k} \rangle_{k=1}^n$ by $\mathbf{x}^\pi$.

Definition 2.4. **ANM** *(Anonymity): Let* π be any permutation of the individuals $(1, ..., n)$, $\langle \succsim_k \rangle_{k=1}^n, \langle \succsim_{\pi_k} \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ *and* $\succsim^\pi = W(\langle \succsim_{\pi_k} \rangle_{k=1}^n)$. Then for every $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{x}^\pi, \mathbf{y}^\pi \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $\mathbf{x}^\pi \succsim^\pi \mathbf{y}^\pi$.

The fifth axiom is a weak version of IIA, in which preference profiles of identical structure with respect to corresponding pairs of social states, ought to lead to an identical social preference between these pairs. Note that only preferences over elementary lotteries are considered in the profiles associated in the axiom.

DEFINITION 2.5. **IIA** (*Independence of Irrelevant Alternatives with Neutral Property): Let* $\langle \succsim_k \rangle_{k=1}^n, \langle \tilde{\succsim}_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$, $\tilde{\succsim} = W(\langle \tilde{\succsim}_k \rangle_{k=1}^n)$ *and* $\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ *such that for each* k, $\mathbf{x} \sim_k p\mathbf{y} \Leftrightarrow \tilde{\mathbf{x}} \tilde{\sim}_k p\tilde{\mathbf{y}}$ and $q\mathbf{x} \sim_k \mathbf{y} \Leftrightarrow q\tilde{\mathbf{x}} \tilde{\sim}_k \tilde{\mathbf{y}}$. *Then* $\mathbf{x} \succsim \mathbf{y}$ *if and only if* $\tilde{\mathbf{x}} \tilde{\succsim} \tilde{\mathbf{y}}$.

The axiom **IIA** is crucial to the characterization of the social welfare function, since it is the only axiom except the domain that involves the origin and its the only one that allows interpersonal comparisons. The axiom involves the probability for which one social state is the certainty equivalent of an elementary lottery that has another social state and $\mathbf{x}^0$. In a sense, this certainty equivalent probability can be seen as a measure for the place of one social state on the scale from the origin to the better state. The axiom says that this kind of measure should matter for the social preference. In other words, if in one profile states $\mathbf{x}$ and $\mathbf{y}$ are related by the same certainty equivalent probability as do states $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ in another profile, then the social ranking for the first profile between $\mathbf{x}$ and $\mathbf{y}$ should be the same as the social ranking for the second profile between $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. This axiom is implied by Arrow's **IIA** but not vice versa. This is the case because this axiom conditions on the information given by the certainty equivalent probabilities, a condition which always implies the condition in the Arrovian **IIA** that only requires that individuals ordinal ranking of states be the same in both profiles. This axiom indeed implies non-comparability for cardinal utilities, since for expected utility preferences for example, the social ranking will depend on utility representations up to positive affine transformations. Instead this axiom suggests an ordinal origin comparability. We can also note that the axiom involves a neutrality property, since it relates different pairs of states in different profiles. This modification is needed as a result of the structure assumed for the set $\mathbf{X}$, without which **IIA** would imply neutrality without assuming it directly, provided that Pareto indifference is assumed.

The first implication of **IIA** is a welfarism result, saying that only utility information is important for the social ranking. However, since we are not necessarily in the expected utility framework, the welfarism result relates to a utility which is different from vNM's. It refers instead to a utility definition that generalizes vNM's utility to non expected utility preferences and relies on the origin. This approach requires the notion of induced utilities, as was first introduced in [7], referring to the Nash bargaining solution. Given a preference relation $\succsim_k \in \mathcal{P}^{ms}$, let $\mathcal{U}_k$ be the set of all continuous functions $u_k : \{x_k \mid \mathbf{x} \in \mathbf{X}\} \times \{x_k \mid \mathbf{x}^0 < \mathbf{x} \in \mathbf{X}\} \to \mathbb{R}_+$ that increase in their first argument, decrease in the second, satisfy $u_k(t;t) = 1$, $u_k(x_k^0;t) = 0$ and $u_k(s;t)\,u_k(t;s) = 1$.

Definition 2.6. Let $k \in \mathbf{N}$. The k^{th} *induced utility mapping* is the function $IU_k : \mathcal{P}^{ms} \to \mathcal{U}_k$, that is defined by

$$IU_k(\succsim_k)(s;t) = \begin{cases} p & \text{if } s \sim_k pt \\ \frac{1}{p} & \text{if } t \sim_k ps \end{cases}.$$

The function $u_k = IU_k(\succsim_k)$ is the *induced utility* of $\succsim_k$. It is easy to see that the induced utilities u_k, whenever they are well defined, satisfy:

$$px_k \sim_k y_k \iff pu_k(x_k;x_k) = u_k(y_k;x_k) \iff pu_k(x_k;y_k) = u_k(y_k;y_k).$$

For any $\succsim_k \in \mathcal{P}^{ms}$ with a multiplicatively separable representation $g_k v_k$, where $v_k(x_k^0) = 0$,

$$u_k(x_k;y_k) = \begin{cases} g_k^{-1}\left(\frac{v_k(x_k)}{v_k(y_k)}\right) & \text{if } y_k \geq x_k \\ \frac{1}{g_k^{-1}\left(\frac{v_k(y_k)}{v_k(x_k)}\right)} & \text{if } x_k > y_k \end{cases},$$

and for EU and DL preference relations, $g_k(p) = p \ \forall p \in [0,1]$, thus $u_k(x_k;y_k) = \frac{v_k(x_k)}{v_k(y_k)}$.

Let $\mathbf{u}_{\langle \succsim_k \rangle_{k=1}^n}(\mathbf{x};\mathbf{y})$ denote the vector $\langle u_k(x_k;y_k)\rangle_{k=1}^n$, where $u_k(\cdot;\cdot)$ are the induced utility functions of $\succsim_k$. The welfarism result for induced utilities is stated as follows (note that this result applies to wider domains than those given by **MS**).

Proposition 2.1. *Let $W(\cdot)$ satisfy* ***IIA*** *on $\mathcal{C} \subseteq \mathcal{P}^n$. Then there exist a set $A_W \subseteq \mathbb{R}_+^n$ such that for any $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{C}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ and $\mathbf{x},\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} > \mathbf{x}^0$, $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \mathbf{u}_{\langle \succsim_k \rangle_{k=1}^n}(\mathbf{x};\mathbf{y}) \in A_W$.*

Proof. Define the set

$$A_W = \{\mathbf{s} \in \mathbb{R}_+^n \mid \exists \langle \succsim_k \rangle_{k=1}^n \in \mathcal{C},\, \mathbf{x},\mathbf{y} \in \mathbf{X} \text{ such that } \mathbf{u}_{\langle \succsim_k \rangle_{k=1}^n}(\mathbf{x};\mathbf{y}) = \mathbf{s},\ \succsim = W(\langle \succsim_k \rangle_{k=1}^n),\, \mathbf{x} \succsim \mathbf{y}\}.$$

For any $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{C}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ and $\mathbf{x},\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} > \mathbf{x}^0$, if $\mathbf{u}_{\langle \succsim_k \rangle_{k=1}^n}(\mathbf{x};\mathbf{y}) \in A_W$, then there exist $\langle \tilde{\succsim}_k \rangle_{k=1}^n \in \mathcal{C}$ and $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$, such that

$\mathbf{u}_{\langle\tilde{\succsim}_k\rangle_{k=1}^n}(\tilde{\mathbf{x}};\tilde{\mathbf{y}}) = \mathbf{u}_{\langle\succsim_k\rangle_{k=1}^n}(\mathbf{x};\mathbf{y})$, $\tilde{\succsim} = W(\langle\tilde{\succsim}_k\rangle_{k=1}^n)$ and $\tilde{\mathbf{x}}\tilde{\succsim}\tilde{\mathbf{y}}$, thus $\mathbf{x} \succsim \mathbf{y}$ by **IIA**. Therefore, for any $\langle\succsim_k\rangle_{k=1}^n \in \mathcal{C}$, $\succsim = W(\langle\succsim_k\rangle_{k=1}^n)$ and any $\mathbf{x},\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} > \mathbf{x}^0$, $\mathbf{u}_{\langle\succsim_k\rangle_{k=1}^n}(\mathbf{x};\mathbf{y}) \in A_W$ holds if, and only if, $\mathbf{x} \succsim \mathbf{y}$. ▌

3. ORDINAL NASH SOCIAL WELFARE FUNCTION FOR SOCIETY OF SIZE TWO

In this section we analyze the axioms implications on the social welfare function, starting by considering the case where $n = 2$, i.e. where only two individuals make up the society. Adding the other axioms to **IIA**, the following result characterizes the set A_W that appear in the welfarism result and consequently the social preference.

PROPOSITION 3.1. *Let $n = 2$ and $W(\cdot)$ satisfy **MS**, **WO**, **PAR**, **ANM** and **IIA**. Then $A_W = \{\mathbf{s} \in \mathbb{R}^2_+ \mid s_1s_2 \geq 1\}$. Furthermore, for any $\langle\succsim_k\rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle\succsim_k\rangle_{k=1}^n)$ and $\mathbf{x},\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\mathbf{y} > \mathbf{x}^0$ implies $u_1(x_1;y_1)u_2(x_2;y_2) \geq 1$.*

Proof. Let $\mathbf{s},\mathbf{t} \in \mathbb{R}^n_+$ such that $\mathbf{s} < \mathbf{t}$. For $\mathbf{s} > \mathbf{0}$, denote $\langle\frac{1}{s_k}\rangle_{k=1}^n$ by $\mathbf{s}^{-1}$. Let $\pi = (2,1)$ denote a permutation of the individuals and denote $\langle s_{\pi_k}\rangle_{k=1}^n$ by $\mathbf{s}^{2,1}$, for any $\langle\succsim_k\rangle_{k=1}^n \in \mathcal{D}$, denote $\langle\succsim_{\pi_k}\rangle_{k=1}^n$ by $\langle\succsim_k^{2,1}\rangle_{k=1}^n$ and for any $\mathbf{z} \in \mathbf{X}$, denote $\langle z_{\pi_k}\rangle_{k=1}^n$ by $\mathbf{z}^{2,1}$. Let $\langle\succsim_k\rangle_{k=1}^n, \langle\succsim_k^{2,1}\rangle_{k=1}^n \in \mathcal{D}$ and let $\succsim = W(\langle\succsim_k\rangle_{k=1}^n)$, $\succsim^{2,1} = W(\langle\succsim_k^{2,1}\rangle_{k=1}^n)$. By continuity and monotonicity of $\succsim_1,\succsim_2$ and $\mathbf{X}$ being $\mathbf{x}^0$-comprehensive, there exist $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbf{X}$ such that $\mathbf{x} < \mathbf{z}$, $\mathbf{y} > \mathbf{x}^0$, $\mathbf{x}^{2,1},\mathbf{y}^{2,1} \in \mathbf{X}$, $\mathbf{s} = \mathbf{u}_{\langle\succsim_k\rangle_{k=1}^n}(\mathbf{x};\mathbf{y})$ and $\mathbf{t} = \mathbf{u}_{\langle\succsim_k\rangle_{k=1}^n}(\mathbf{z};\mathbf{y})$. Therefore $\mathbf{s}^{2,1} = \mathbf{u}_{\langle\succsim_k^{2,1}\rangle_{k=1}^n}(\mathbf{x}^{2,1};\mathbf{y}^{2,1})$ and $\mathbf{s} > \mathbf{0}$ implies $\mathbf{s}^{-1} = \mathbf{u}_{\langle\succsim_k\rangle_{k=1}^n}(\mathbf{y};\mathbf{x})$ and $\mathbf{t}^{-1} = \mathbf{u}_{\langle\succsim_k\rangle_{k=1}^n}(\mathbf{y};\mathbf{z})$. By **ANM,** $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\mathbf{x}^{2,1} \succsim^{2,1} \mathbf{y}^{2,1}$, thus $\mathbf{s} \in A_W$ if, and only if, $\mathbf{s}^{2,1} \in A_W$. Furthermore $\mathbf{0} < \mathbf{s} \notin A_W$ implies $\mathbf{s}^{-1} \in A_W$ by completeness of $\succsim$. Moreover, $\mathbf{s} \in A_W$ implies $\mathbf{t} \in A_W$ and $\mathbf{t}^{-1} \notin A_W$ since $\mathbf{z} \succ \mathbf{x} \succsim \mathbf{y}$ by **PAR** and therefore $\mathbf{z} \succ \mathbf{y}$ by transitivity.
Next, we will show that $\mathbf{s} \in A_W$ if, and only if, $s_1s_2 \geq 1$. Suppose first that $\mathbf{s},\mathbf{s}^{2,1} \in A_W$. If $0 < s_1s_2 < 1$, then $\mathbf{s}^{-1} > \mathbf{s}^{2,1}$ since $(\mathbf{s}^{-1})_1 = \frac{1}{s_1} > s_2 = (\mathbf{s}^{2,1})_1$ and $(\mathbf{s}^{-1})_2 = \frac{1}{s_2} > s_1 = (\mathbf{s}^{2,1})_2$. Therefore $\mathbf{s}^{-1} \in A_W$ and $\mathbf{s} \notin A_W$, a contradiction. If $s_1s_2 = 0$, then there exist $\tilde{\mathbf{s}} > \mathbf{s}$ such that $0 < \tilde{s}_1\tilde{s}_2 < 1$, thus $\tilde{\mathbf{s}} \in A_W$, again a contradiction. Hence, $\mathbf{s} \in A_W$ implies $s_1s_2 \geq 1$. To prove the converse, suppose that $s_1s_2 \geq 1$. If $s_1s_2 = 1$, then $\mathbf{s}^{-1} = \mathbf{s}^{2,1}$, thus $\mathbf{s},\mathbf{s}^{2,1} \in A_W$. If $s_1s_2 > 1$, then $\mathbf{s} \in A_W$, otherwise $\mathbf{s}^{-1} \in A_W$, thus $s_1s_2 \leq 1$, a contradiction. Hence $\mathbf{s} \in A_W$ if, and only if, $s_1s_2 \geq 1$. Therefore for any $\langle\succsim_k\rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle\succsim_k\rangle_{k=1}^n)$ and $\mathbf{x},\mathbf{y} \in \mathbf{X}$, it follows that $\mathbf{y} > \mathbf{x}^0$ implies $\mathbf{x} \succsim \mathbf{y} \iff u_i(x_i;y_i)u_j(x_j;y_j) \geq 1$ by proposition 2.1. Furthermore, not $\mathbf{y} > \mathbf{x}^0$ implies $\mathbf{x} \succsim \mathbf{y}$. Otherwise,

assuming $\mathbf{y} \succ \mathbf{x}$, **PAR** implies not $\mathbf{x} > \mathbf{y}$, therefore there exist $p \in [0,1]$ satisfying $\mathbf{y} \sim_i 0\mathbf{x}$ and $p\mathbf{y} \sim_j \mathbf{x}$, thus for any $\mathbf{z} \in \mathbf{X}$ such that $\mathbf{z} \succ_i \mathbf{x}^0$, $\mathbf{x}^0 \sim_i 0\mathbf{z}$, $\mathbf{z} \sim_j \mathbf{x}^0$ and $p\mathbf{x}^0 \sim_j \mathbf{z}$, it follows that $\mathbf{x}^0 \succ \mathbf{z}$ by **IIA**, contradicting **PAR**. Hence for any $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\mathbf{y} > \mathbf{x}^0$ implies $u_i(x_i; y_i) u_j(x_j; y_j) \geq 1$. ∎

The proposition leads to the definition of a social preference which takes into account the relative position of states with respect to the worst state, $\mathbf{x}^0$. The relative position is defined by the probability, according to which a mixture of one state combined with $\mathbf{x}^0$ is equivalent to a second state. In general, suppose for the pair of states $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, that $p\mathbf{x} \sim_1 \mathbf{y}$, $q\mathbf{y} \sim_2 \mathbf{x}$ and $p \leq q$. In this case, we understand that individual 1 prefers state $\mathbf{x}$ over state $\mathbf{y}$ and individual 2 has opposing preferences, but $\mathbf{x}$ is located in a higher position for 2 than $\mathbf{y}$ is for 1, relative to the worst state $\mathbf{x}^0$. In other words, the transition for each individual from her preferred state to the other state is more costly for 1 than it is for 2. Thus, we could argue that society should prefer $\mathbf{x}$, the strongly preferred state for individual 1, over the state $\mathbf{y}$. According to this interpretation, individual 1's preference for state $\mathbf{x}$ over state $\mathbf{y}$ is more intense than individual 2's opposite preference. In the case of Pareto dominance, where such a comparison is not possible, the preference relation is defined to agree with the Pareto rule, i.e. the dominating state is preferred.

We refer to the social preference relation defined in this way as the '*ordinal Nash social preference relation*'. The choice of name for the social preference is justified, as shown below, by its connection to the Nash social preference relation, which is defined for EU preference profiles according to the product of vNM utility gains above the worst state $\mathbf{x}^0$. The function that assigns to every preference profile of the society, the corresponding ordinal Nash social preference relation, when it is well defined, is called the '*ordinal Nash social welfare function*'.

Following the above interpretation we define:

Definition 3.1. The ordinal Nash social welfare function, denoted $ON(\cdot)$, assigns for any preference relation profile $\langle \succsim_1, \succsim_2 \rangle \in (\mathcal{P}^{ms})^2$, a preference relation $\succsim \subseteq \mathbf{X}^2$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if either $\mathbf{x} \succsim_k \mathbf{y}$ for both k or there exist $i, j \in \mathbf{N}$ and $p, q \in [0,1]$ such that $p\mathbf{x} \sim_i \mathbf{y}$, $q\mathbf{y} \sim_j \mathbf{x}$ and $p \leq q$.

Note, that the social preference relation $\succsim$ is defined only on the set of social states $\mathbf{X}$ and does not compare social lotteries in the set $\mathcal{L}$. Thus, definition 3.1 only utilizes information about individual preferences on the set $\mathcal{E}$ of elementary lotteries. In order to analyze the preference relation definition in depth, we state lemma 3.1, which defines a convenient property of the binary relation $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$ in connection to induced utilities and shows that the binary relation is implied by the axioms.

Lemma 3.1. *Let $\langle \succsim_1, \succsim_2 \rangle \in (\mathcal{P}^{ms})^2$ and $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$, where $u_k(\cdot;\cdot)$ are the induced utility functions of $\succsim_k$. Then for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\mathbf{y} > \mathbf{x}^0$ implies $u_i(x_i; y_i) u_j(x_j; y_j) \geq 1$. Furthermore, not $\mathbf{x} > \mathbf{x}^0$ and $\mathbf{x} \succsim \mathbf{y}$ implies not $\mathbf{y} > \mathbf{x}^0$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Then either $\mathbf{x} \succsim_k \mathbf{y}$ for both $k \in \mathbf{N}$ or there exist $i, j \in \mathbf{N}$ and $p, q \in [0,1]$ such that $p\mathbf{x} \sim_i \mathbf{y}$, $q\mathbf{y} \sim_j \mathbf{x}$, where by definition $p = u_i(y_i; x_i)$, $q = u_j(x_j; y_j)$. Therefore $\mathbf{y} > \mathbf{x}^0$ implies $\mathbf{x} \succsim \mathbf{y} \iff \prod_k u_k(x_k; y_k) \geq 1$. If not $\mathbf{y} > \mathbf{x}^0$ then $\mathbf{x} \succsim \mathbf{y}$ since either $\mathbf{x} \succsim_k \mathbf{y}$ for both $k \in \mathbf{N}$ or $0 = p \leq q$. Therefore, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\mathbf{y} > \mathbf{x}^0$ implies $u_i(x_i; y_i) u_j(x_j; y_j) \geq 1$. If not $\mathbf{x} > \mathbf{x}^0$ and $\mathbf{x} \succsim \mathbf{y}$ then not $\mathbf{y} > \mathbf{x}^0$, otherwise $\prod_k u_k(x_k; y_k) = 0$ and thus not $\mathbf{x} \succsim \mathbf{y}$, a contradiction. ∎

For any preference profile $\langle \succsim_1, \succsim_2 \rangle \in (\mathcal{P}^{ms})^2$, for which the relation $\succsim$ is complete and transitive, it can be considered a well defined, social preference. Therefore, corollary 3.1 is meaningful.

Corollary 3.1. *Let $D^C \subseteq (\mathcal{P}^{ms})^2$ be the set of all $\langle \succsim_1, \succsim_2 \rangle$ for which $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$ is a complete binary relation. Then,*
(1) $D^C = (\mathcal{P}^{ms})^2$.
(2) If $\succsim_i$ are EU or DL, then $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$ is also transitive.

Proof. (1) According to the induced utilities definition, for any $\mathbf{x}^0 < \mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $k \in \mathbf{N}$, $u_k(x_k; y_k) u_k(y_k; x_k) = 1$. Thus either $\prod_k u_k(x_k; y_k) \geq 1$ or $\prod_k u_k(y_k; x_k) \geq 1$ and therefore $\succsim$ is complete by lemma 3.1.
(2) If $\succsim_i$ are EU or DL, then for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ where $\mathbf{y} > \mathbf{x}^0$ and $k \in \mathbf{N}$, $u_k(x_k; y_k) = \frac{v_k(x_k)}{v_k(y_k)}$. Let $\mathbf{x} \succsim \mathbf{y} \succsim \mathbf{z}$. If not $\mathbf{y} > \mathbf{x}^0$ or not $\mathbf{z} > \mathbf{x}^0$, then $\mathbf{x} \succsim \mathbf{z}$ by lemma 3.1. Otherwise, $\prod_k u_k(x_k; z_k) = \prod_k \frac{v_k(x_k)}{v_k(z_k)} = \prod_k \frac{v_k(x_k)}{v_k(y_k)} \frac{v_k(y_k)}{v_k(z_k)} = \prod_k u_k(x_k; y_k) \prod_k u_k(y_k; z_k) \geq 1$ by lemma 3.1, therefore $\mathbf{x} \succsim \mathbf{z}$. ∎

The corollary states that for every profile where preferences are EU or DL, the social preference is well defined. Unfortunately, there exist examples of preference profiles $\langle \succsim_1, \succsim_2 \rangle \in (\mathcal{P}^{ms})^2$, for which $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$ is not transitive.

Example 3.1. Let $\langle \succsim_1, \succsim_2 \rangle \in (\mathcal{P}^{ms})^2$, where $\succsim_1$ is a risk neutral EU preference with a vNM utility function $v_1(x_1) = x_1$ and $\succsim_2$ is a DA preference represented by $g_2 v_2$ such that $v_2(x_2) = x_2$ and $g_2(p) = \frac{p}{1+(1-p)\beta_2}$, $\beta_2 = 1$. The corresponding induced utility functions satisfy $u_1(x_1; y_1) = \frac{x_1}{y_1}$

and $u_2(x_2;y_2) = \frac{2}{1+y_2/x_2}$ for $\mathbf{y} \succsim_2 \mathbf{x}$. Let $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^2_+ \mid \sum_k x_k \leq 9\}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, where $\mathbf{x} = (6,1)$, $\mathbf{y} = (3,3)$, $\mathbf{z} = (2,6)$ and $\mathbf{x}^0 = (0,0)$. Then, $\prod_k u_k(x_k;y_k) = \frac{6}{3}\frac{2}{1+3/1} = 1$, $\prod_k u_k(y_k;z_k) = \frac{3}{2}\frac{2}{1+6/3} = 1$ and $\prod_k u_k(x_k;z_k) = \frac{6}{2}\frac{2}{1+6/1} < 1$. Therefore, by lemma 3.1, $\mathbf{x} \sim \mathbf{y} \sim \mathbf{z}$ but $\mathbf{x} \prec \mathbf{z}$, as demonstrated in the figure 1.

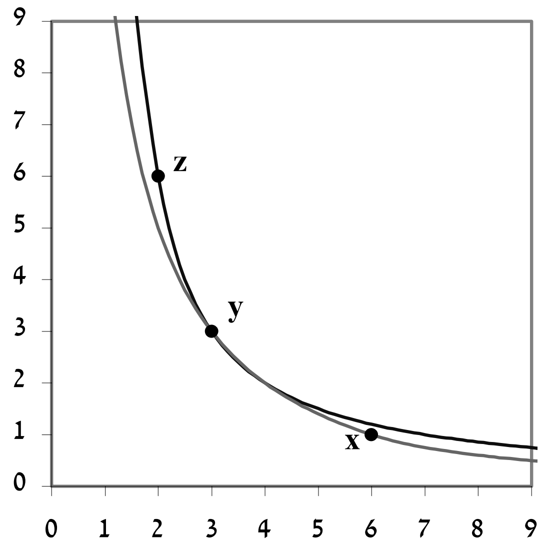

Figure 1: Non-transitive social preference indifference curves

This example motivates the investigation of sets of multiplicatively separable preference profiles on which $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$ represents a transitive binary relation over the set $\mathbf{X}$. In this example, the individuals have different probability distortion functions, one of them fits the expected utility assumption while the other does not. It turns out that this fact is crucial to transitivity, as can be seen by proposition 3.2. The result proves a necessary and sufficient condition for transitivity, stated on the representation functions corresponding to any preference profile in $(\mathcal{P}^{ms})^2$.

Proposition 3.2. *Let $\langle \succsim_1, \succsim_2 \rangle \in (\mathcal{P}^{ms})^2$ and $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$. Then, $\succsim$ is transitive if, and only if, $\succsim_k$ $(k \in \mathbf{N})$ have multiplicatively separable representations $g_k v_k$, such that $g_1 = g_2$.*

Proof. Let $u_k(\cdot;\cdot)$ be the induced utility functions of $\succsim_k$. Suppose there exist functions g, v_1, v_2 such that for both k, $\succsim_k$ can be represented for any $p\mathbf{x} \in \mathcal{E}$ by $g(p)v_k(x_k)$, where $v_k(x^0_k) = 0$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that $\mathbf{x} \succsim \mathbf{y} \succsim \mathbf{z}$. If $\mathbf{x} \geq \mathbf{z}$ or not $\mathbf{x} > \mathbf{x}^0$ or not $\mathbf{y} > \mathbf{x}^0$ or not $\mathbf{z} > \mathbf{x}^0$, then $\mathbf{x} \succsim \mathbf{z}$ by lemma 3.1. Otherwise, by lemma 3.1, either $u_k(x_k;y_k) \geq 1$

for both $k \in \mathbf{N}$ or there exist $i \in \mathbf{N}$ such that $g^{-1}(\frac{v_i(y_i)}{v_i(x_i)}) = u_i(y_i; x_i) \leq u_j(x_j; y_j) = g^{-1}(\frac{v_j(x_j)}{v_j(y_j)})$, thus $\prod_k \frac{v_k(x_k)}{v_k(y_k)} \geq 1$. Similarly, $\prod_k \frac{v_k(y_k)}{v_k(z_k)} \geq 1$, therefore $\prod_k \frac{v_k(x_k)}{v_k(z_k)} \geq 1$. Let $\hat{\imath}, \hat{\jmath} \in \mathbf{N}$ such that $\mathbf{z} \succsim_{\hat{\imath}} \mathbf{x}$ and $\mathbf{x} \succsim_{\hat{\jmath}} \mathbf{z}$. Then, $u_{\hat{\imath}}(x_{\hat{\imath}}; z_{\hat{\imath}}) = g^{-1}(\frac{v_{\hat{\imath}}(x_{\hat{\imath}})}{v_{\hat{\imath}}(z_{\hat{\imath}})}) \geq g^{-1}(\frac{v_{\hat{\jmath}}(z_{\hat{\jmath}})}{v_{\hat{\jmath}}(x_{\hat{\jmath}})}) = u_{\hat{\jmath}}(z_{\hat{\jmath}}; x_{\hat{\jmath}})$, thus $\mathbf{x} \succsim \mathbf{z}$ by lemma 3.1. Hence $\succsim$ is transitive.

To prove the converse, suppose that for both k, $\succsim_k$ is represented for any $p\mathbf{x} \in \mathcal{E}$ by $g_k(p)v_k(x_k)$, where $v_k(x_k^0) = 0$. Fix $k = 2$ and for any $p, q \in [0,1]$, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ satisfy $g_k^{-1}(\alpha_1) = g_1^{-1}(\alpha_2) = p$ and $g_k^{-1}(\beta_1) = g_1^{-1}(\beta_2) = q$. Then, by continuity and monotonicity of $v_k(\cdot)$, $v_k(x_k^0) = 0$ and $\mathbf{X}$ being $\mathbf{x}^0$-comprehensive, there exist $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that $\alpha_1 = \frac{v_k(z_k)}{v_k(y_k)}$, $\alpha_2 = \frac{v_1(y_1)}{v_1(z_1)}$, $\beta_1 = \frac{v_k(y_k)}{v_k(x_k)}$ and $\beta_2 = \frac{v_1(x_1)}{v_1(y_1)}$. Therefore $u_k(z_k; y_k) = g_k^{-1}(\alpha_1) = g_1^{-1}(\alpha_2) = u_1(y_1; z_1)$ and $u_k(y_k; x_k) = g_k^{-1}(\beta_1) = g_1^{-1}(\beta_2) = u_1(x_1; y_1)$, thus $\mathbf{x} \sim \mathbf{y} \sim \mathbf{z}$. Then, transitivity implies $\mathbf{x} \sim \mathbf{z}$, thus $g_k^{-1}(\alpha_1\beta_1) = u_k(z_k; x_k) = u_1(x_1; z_1) = g_1^{-1}(\alpha_2\beta_2)$. It follows that for any $p, q \in [0,1]$,

$$g_k^{-1}[g_k(p)g_k(q)] = g_1^{-1}[g_1(p)g_1(q)].$$

Fix $p \in (0,1)$ and let $h : \mathbb{R}_+ \to [0,1]$ be defined such that for every $\gamma > 0$, $g_k(h_\gamma) = [g_k(p)]^\gamma$. We will show that for every $\gamma > 0$, $g_1[h_\gamma] = [g_1(p)]^\gamma$. For any positive integers $l, m \geq 1$, let $\gamma_{l,m} = 2^{1-l}m$. Note that for any $\gamma > 0$, there exists a sequence $\{(l_s, m_s)\}_{s=1}^{\infty}$ such that $\lim_{s\to\infty} \gamma_{l_s,m_s} = \gamma$. Therefore since g_1, h are continuous functions, we can restrict attention to $\gamma_{l,m}$ for all integers $l, m \geq 1$. Then for $l \geq 2$ and $m = 1$, $g_k(h_{\gamma_{l-1,1}}) = [g_k(p)]^{\gamma_{l-1,1}} = [[g_k(p)]^{\gamma_{l,1}}]^2 = [g_k(h_{\gamma_{l,1}})]^2$, thus $h_{\gamma_{l-1,1}} = g_k^{-1}[[g_k(h_{\gamma_{l,1}})]^2] = g_1^{-1}[[g_1(h_{\gamma_{l,1}})]^2]$, therefore $g_1(h_{\gamma_{l,1}}) = [g_1(h_{\gamma_{l-1,1}})]^{0.5}$. Since $h_1 = p$, then for $l = 1$, $g_1[h_{\gamma_{1,1}}] = [g_1(p)]^{\gamma_{1,1}}$, thus by induction for any $l \geq 1$,

$$g_1(h_{\gamma_{l,1}}) = [[g_1(p)]^{\gamma_{l-1,1}}]^{0.5} = [g_1(p)]^{\gamma_{l,1}}.$$

For any $l, m \geq 2$, $g_k(h_{\gamma_{l,m}}) = g_k(h_{\gamma_{l,1}})\ g_k(h_{\gamma_{l,m-1}})$, therefore $g_1(h_{\gamma_{l,m}}) = g_1[g_k^{-1}[\ g_k(h_{\gamma_{l,1}})\ g_k(h_{\gamma_{l,m-1}})]] = g_1(h_{\gamma_{l,1}})\ g_1(h_{\gamma_{l,m-1}})$. For $m = 1$, $g_1(h_{\gamma_{l,1}}) = [g_1(p)]^{\gamma_{l,1}}$, thus it follows again by induction that for any $l, m \geq 1$, $g_1(h_{\gamma_{l,m}}) = [g_1(p)]^{\gamma_{l,1}}[g_1(p)]^{\gamma_{l,m-1}} = [g_1(p)]^{\gamma_{l,m}}$. Hence, by continuity of g_1 and h, for every $\gamma > 0$,

$$g_1[h_\gamma] = [g_1(p)]^\gamma.$$

Since $g_k^{-1}[[g_k(p)]^\gamma] = g_k^{-1}[g_k(h_\gamma)] = h_\gamma = g_1^{-1}[g_1(h_\gamma)] = g_1^{-1}[[g_1(p)]^\gamma]$, it follows that for $\delta_k(p) = \frac{\log g_1(p)}{\log g_k(p)} > 0$ and every $\beta \in [0,1]$, $g_k^{-1}[\beta] = g_1^{-1}[\beta^{\delta_k(p)}]$, thus for every $q \in [0,1]$, $g_1(q) = [g_k(q)]^{\delta_k(p)}$. Furthermore, for any $p' \in$

$(0,1)$ and $q \in [0,1]$, $g_1(q) = [g_k(q)]^{\delta_k(p')}$, thus $\delta_k(p) = \delta_k(p')$. Therefore, for every $p \in (0,1)$, $\delta_k(p)$ is constant and its value is denoted δ_k. Hence, for every $p \in [0,1]$, $g_1(p) = [g_k(p)]^{\delta_k}$. Define the function $\tilde{v}_k$ such that $\tilde{v}_k(x_k) = [v_k(x_k)]^{\delta_k}$, thus $\tilde{v}_k(x_k^0) = 0$. It follows that $\succsim_k$ can be represented for any $p\mathbf{x} \in \mathcal{E}$ by $g_1(p)\tilde{v}_k(x_k)$. ▌

Proposition 3.2 shows that expected utility is not necessary to derive transitivity of the *ON* social preference. We can have profiles of non-EU preferences, as long as all preferences violate the expected utility assumption in a similar way. Note that the result permits the characterization of transitivity through a condition dependent only on the distortion functions g_k in the representation corresponding to $\succsim_k$, not on the functions v_k. This separation is permissible due to the multiplicatively separable property of the preference relations in $\mathcal{P}^{ms}$. Note that the necessity of the condition is a direct result of the requirement that the social indifference relation $\sim$ is transitive. Nevertheless, this condition is sufficient for the transitivity of the whole social relation $\succsim$.

Based on proposition 3.2 we may define the following uniform distortion property and the corresponding subset of profiles in $(\mathcal{P}^{ms})^n$ which satisfy this property. Definition 3.2 applies to the general case where $n \geq 2$.

Definition 3.2. **UD** (Uniform Distortion): A preference relation profile $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{ms})^n$ satisfies **UD** if there exist multiplicatively separable representations $g_k v_k$ $(k \in \mathbf{N})$ such that $g_i = g_j = g$ for every $i, j \in \mathbf{N}$, i.e. for every $p\mathbf{x}, q\mathbf{y} \in \mathcal{E}$, $p\mathbf{x} \succsim_k q\mathbf{y} \Leftrightarrow g(p)v_k(x_k) \geq g(q)v_k(y_k)$.

Definition 3.3. (1) Let $\mathcal{G}$ be the set of all functions $g : [0,1] \to [0,1]$ which are onto, continuous and monotonic.
(2) For every $g \in \mathcal{G}$, define $\mathcal{P}^{ms}(g) \subseteq \mathcal{P}^{ms}$ to be the set of all preference relations that have a multiplicatively separable representation gv for some function v.
(3) For every $g \in \mathcal{G}$, define $\mathcal{D}^{UD}(g) = (\mathcal{P}^{ms}(g))^n$.
(4) Let $\mathcal{D}^{UD} = \cup_{g \in \mathcal{G}} \mathcal{D}^{UD}(g)$.

For any $g \in \mathcal{G}$, the set $\mathcal{D}^{UD}(g)$ includes all preference profiles in $(\mathcal{P}^{ms})^n$ for which all individual preference relations have multiplicatively separable representations with the same probability distortion function g. Note however, that the union set, $\mathcal{D}^{UD}$, of the sets $\mathcal{D}^{UD}(g)$ over all the functions $g \in \mathcal{G}$ does not equal $(\mathcal{P}^{ms})^n$. In other words, the sets $\mathcal{D}^{UD}(g)$, $\forall g \in \mathcal{G}$ are not a partition of the set $(\mathcal{P}^{ms})^n$. As a result, there are profiles in $(\mathcal{P}^{ms})^n \backslash \mathcal{D}^{UD}$ which do not satisfy the **UD** property and therefore the *ON* social preference in not transitive for these profiles, as shown in example 3.1.

Following are examples for known families of preferences for which the intersection with the set $\mathcal{D}^{UD}$ is non empty.

Example 3.2. (1) EU and DL preferences. The set $\mathcal{D}^{UD}(g)$ where $g(p) = p$, $\forall p \in [0,1]$ is the set of DL preference profiles, that contains the set of EU preference profiles.
(2) RDU preferences. The set $\mathcal{D}^{UD}(g)$ contains the set of all RDU preference profiles for which the distortion functions are g^{δ_k}, where $\delta_k > 0$, $\forall k \in \mathbf{N}$. In other words, the individual distortion functions are the same up to a positive power transformation. This is the case since for any k^{th} RDU preference with representation $[g(p)]^{\delta_k} v_k(x_k)$ for elementary lotteries $p\mathbf{x} \in \mathcal{E}$, the function $g(p)[v_k(x_k)]^{1/\delta_k}$ also represents the preference for $p\mathbf{x} \in \mathcal{E}$. Taking a positive power transformation of a rank dependent function g preserves the **UD** property for elementary lotteries, but not for general lotteries. Therefore, the kind of restriction implemented by **UD** is in a sense weak, since only preferences over elementary lotteries are involved. The distortion function g is used in the RDU preference representation by computing the distorted expected value with respect to general lotteries, so implementation of **UD** for general lotteries would be much more restrictive[3].
(3) DA preferences. The set $\mathcal{D}^{UD}(g)$ where $g(p) = \frac{p}{1+(1-p)\beta}$ for some $\beta > 0$ contains all DA preference profiles with representations that have equal parameters $\beta_k = \beta$, $\forall k \in \mathbf{N}$. Thus, the set $\mathcal{D}^{UD}$ contains all DA preference profiles for which the individual parameters β_k are equal (see example 3.1 for the case of non transitivity under which this condition is violated).

Proposition 3.2 states that the *ON* social preference is transitive only for all the profiles in the set $\mathcal{D}^{UD}$. This seems to be a disappointing result, since it limits the applicability of the social aggregation rule implied by the axioms. But it also has some interesting positive implications. The first one concerns the role of the expected utility assumption in the derivation of the Nash social welfare function. It turns out that the axioms that imply the social aggregation rule are consistent in the case of expected utility preferences only because these preferences have a common feature with respect to risk attitude. The important issue is the initial agreement, to some extent, on risk attitudes, not the specific way this agreement is characterized. Whether it is expected utility or some other assumption on

[3]In the case of expected utility this observation has no implication since uniform distortion holds for general lotteries by not being distorted at all.

risk attitude, the crucial factor is that all individuals agree on it. This conclusion suggests that the expected utility assumption is too restrictive in the analysis of the connection between individual and social choice.

Another implication of this result concerns the representation of the social preference, as shown by theorem 3.1. The theorem states that the function $ON(\cdot)$ is well defined and characterized uniquely by the axioms only on uniformly distorted domains and moreover has a Nash-like utility product representation.

THEOREM 3.1. *(1) Let* $\langle \succsim_1, \succsim_2 \rangle \in \mathcal{D}^{UD}$*, where* $\succsim_k$ *have multiplicatively separable representations* gv_k*, and let* $\succsim = ON(\langle \succsim_1, \succsim_2 \rangle)$*. Then, for any* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$*,* $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \prod_k v_k(x_k) \geq \prod_k v_k(y_k)$*.*
(2) Let $W(\cdot)$ *satisfy* ***MS*****,** ***WO*****,** ***PAR*****,** ***ANM*** *and* ***IIA****. Then* $W = ON$ *defined over* $\mathcal{D} \subseteq \mathcal{D}^{UD}$*, e.g* $\mathcal{D} = \mathcal{D}^{UD}$*.*

Proof. (1) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $u_k(\cdot;\cdot)$ be the induced utility functions of $\succsim_k$. If not $\mathbf{y} > \mathbf{x}^0$, then $\mathbf{x} \succsim \mathbf{y}$ and $\prod_i v_i(x_i) \geq \prod_i v_i(y_i) = 0$. If $\mathbf{x}, \mathbf{y} > \mathbf{x}^0$ then by lemma 3.1, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, either $v_k(x_k) \geq v_k(y_k)$ for both $k \in \mathbf{N}$ or there exist $i \in \mathbf{N}$ such that $g^{-1}(\frac{v_i(y_i)}{v_i(x_i)}) \leq g^{-1}(\frac{v_j(x_j)}{v_j(y_j)})$. Therefore, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$.
(2) This follows immediately from axioms consistency, propositions 3.1 and 3.2 and lemma 3.1. We now prove consistency of the axioms by showing that $ON(\cdot)$ satisfies them. Transitivity and completeness follow from proposition 3.2 and corollary 3.1. If $\mathbf{x} \succsim_k \mathbf{y}$ for each k and $\mathbf{x} \succ_j \mathbf{y}$ for some j, then $\prod_k v_k(x_k) > \prod_k v_k(y_k)$ and therefore $\mathbf{x} \succ \mathbf{y}$, hence **PAR** is satisfied. Axiom **ANM** is satisfied since for any permutation π of the players and any $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x}^\pi \in \mathbf{X}$, $\prod_k v_{\pi_k}(x_{\pi_k}) = \prod_k v_k(x_k)$. Checking axiom **IIA**, $\mathbf{x} \succsim \mathbf{y} > \mathbf{x}^0$ implies $\prod_k u_k(x_k; y_k) \geq 1$ by lemma 3.1 and $u_k(x_k; y_k) = \tilde{u}_k(\tilde{x}_k; \tilde{y}_k)$ for each $k \in \mathbf{N}$ implies $\prod_k \tilde{u}_k(\tilde{x}_k; \tilde{y}_k) \geq 1$ and therefore $\tilde{\mathbf{x}} \tilde{\succsim} \tilde{\mathbf{y}}$. If not $\mathbf{y} > \mathbf{x}^0$, then not $\tilde{\mathbf{y}} > \mathbf{x}^0$, thus $\tilde{\mathbf{x}} \tilde{\succsim} \tilde{\mathbf{y}}$. Hence **IIA** is satisfied. ∎

Note that the completeness and transitivity of $\succsim$ agrees with the completeness and transitivity of the relation $\geq$ on the set $\{\prod_k v_k(x_k) \in \mathbb{R}_+ | \ \mathbf{x} \in \mathbf{X}\}$.

Utilizing the ON social welfare function, in the case of EU preferences, theorem 3.1 provides an interpretation of Nash's utility product maximization principle for social choice from a set of deterministic social states $\mathbf{X}$. The theorem also extends this interpretation by considering non-expected utility preference profiles in $\mathcal{P}^{ms}$. Furthermore, this extension yields the largest subset of $(\mathcal{P}^{ms})^2$ for which the interpretation is valid, i.e. for which the ON social welfare function is well defined. For this extended domain, the theorem states that information provided by the value functions v_k, corresponding to the preference relations $\succsim_k$, is sufficient in its own right to derive the social preference. This conclusion holds despite the fact that the value functions v_k generally convey only partial information about the preferences in $\mathcal{P}^{ms}$ over the set of elementary lotteries $\mathcal{E}$.

4. ORDINAL NASH SOCIAL WELFARE FUNCTION FOR SOCIETY OF SIZE GREATER THAN TWO

In the general case, where the society has more than two individuals, the *ON* interpretation can be extended to allow comparisons between social states that are different for more than two individuals. Such a comparison is not as clear as in the case of $n = 2$. For example, in the case where $n = 3$, a social state may be preferable for one individual but worse for the other two, thus a direct comparison as in the case where $n = 2$ is not possible. In this scenario, the use of the majority rule for example, would not express the intensity of preference as considered in the former definition. In order to incorporate such comparisons, we extend the social preference definition by utilizing its transitivity property when a direct comparison is not possible, as implied by the axioms.

Definition 4.1. Given a profile $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{ms})^n$, for any $l = 1, 2, ...$, define the preference relations $\succsim^l \subseteq \mathbf{X}^2$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$:
(1) $\mathbf{x} \succsim^1 \mathbf{y}$ if there exist $i, j \in \mathbf{N}$ such that $\mathbf{x} \sim_k \mathbf{y}$ for each $k \in \mathbf{N}\backslash\{i, j\}$ and either $\mathbf{x} \succsim_k \mathbf{y}$ for $k \in \{i, j\}$ or there exist $p, q \in [0, 1]$ such that $p\mathbf{x} \sim_i \mathbf{y}$, $q\mathbf{y} \sim_j \mathbf{x}$ and $p \leq q$.
(2) For any $l \geq 2$, $\mathbf{x} \succsim^l \mathbf{y}$ if there exist $\mathbf{z} \in \mathbf{X}$ such that $\mathbf{x} \succsim^{l-1} \mathbf{z} \succsim^1 \mathbf{y}$.

Definition 4.2. The ordinal Nash social welfare function, denoted $ON(\cdot)$, assigns for any preference relation profile $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{ms})^n$, a preference relation $\succsim \subseteq \mathbf{X}^2$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if
(1) $\mathbf{x} \succsim^1 \mathbf{y}$ or,
(2) $\mathbf{x} \succsim^l \mathbf{y}$ for $l > 1$ and $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} > 2$.

This definition, despite being long, is a straightforward generalization of the previous one for $n = 2$. The previous definition 3.1 is provided by $\succsim^1$, except that all individuals other than the two which are involved in the direct comparison must be indifferent between the states being compared. For $\succsim^l$ where $l \geq 2$, we get a preference that takes into account higher transitivity implications of $\succsim^1$. Note, that when $n = 2$ or when the social states being compared are different for two individuals alone, the extended definition coincides with the previous one. In all other cases, the comparison is carried out indirectly through a sequence of comparisons as in the

$n = 2$ scenario. An illustration of an indirect comparison is presented in example 4.1.

Example 4.1. Let $n = 3$, $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \sum_k x_k \leq 9\}$, $\mathbf{x}^0 = (0,0,0)$, $\mathbf{x} = (3,3,3)$, $\mathbf{y} = (1,4,4)$ and let $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{ms})^n$, where $\succsim_k$ are DL preferences with a value function $v_k(x_k) = x_k$ and $g_k(p) = p \; \forall p \in [0,1]$. Then $\mathbf{x} \succsim^1 \mathbf{y}$ does not hold since $\mathbf{x}, \mathbf{y}$ are different for all $k \in \mathbf{N}$. Nevertheless, $\mathbf{z} = (2,4,3) \in \mathbf{X}$ satisfies $\mathbf{x} \succ^1 \mathbf{z}$ since $\mathbf{x}, \mathbf{z}$ are different only for individuals 1 and 2, $\frac{2}{3}\mathbf{x} \sim_1 \mathbf{z}$, $\frac{3}{4}\mathbf{z} \sim_2 \mathbf{x}$ and $\frac{2}{3} < \frac{3}{4}$. Similarly, $\mathbf{z} \succ^1 \mathbf{y}$ since $\mathbf{z}, \mathbf{y}$ are different only for individuals 1 and 3, $\frac{1}{2}\mathbf{z} \sim_1 \mathbf{y}$, $\frac{3}{4}\mathbf{y} \sim_3 \mathbf{z}$ and $\frac{1}{2} < \frac{3}{4}$. Thus, $\mathbf{x} \succsim^2 \mathbf{y}$ and therefore $\mathbf{x} \succsim \mathbf{y}$. Note that in order to deduce $\mathbf{x} \succ \mathbf{y}$ without assuming transitivity of $\succsim$ directly, we must show that $\mathbf{y} \succsim^l \mathbf{x}$ does not hold for any $l \geq 2$.

When the number of individuals that care about the choice between two states is 2, the social preference definition requires a direct comparison as defined by $\succsim^1$. In other words, transitivity implications are included in the definition only when direct comparison is not possible. Although the *ON* social preference definition is extended based on a sequence of comparisons, thus ensuring transitivity, this property does not necessarily hold if the social states being compared are different for two individuals only. The example of non transitivity given in section 3 is also relevant here if we assume that all individuals except two are indifferent between the states in the example. Furthermore, in contrast to the case where $n = 2$, the extended social preference is not necessarily complete, since the existence of a sequence which permits indirect comparison is not guaranteed for any pair of social states.

Hence, we are interested in conditions that enable the *ON* social preference to be well defined. As before, the induced utilities are useful in this analysis. Lemma 4.1, which is an analogue of lemma 3.1, translates definition 4.2 in terms of induced utilities (proof in the appendix).

Lemma 4.1. *Let* $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{ms})^n$ *and* $\succsim = ON(\langle \succsim_k \rangle_{k=1}^n)$, *where* $u_k(\cdot;\cdot)$ *are the induced utility functions of* $\succsim_k$. *Then, for any* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$:
(1) The following are equivalent:
(a) $\mathbf{x} \succsim \mathbf{y}$
(b) There exist $m \geq 2$ *and sequences* $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m, \{i_l \in \mathbf{N}\}_{l=2}^m, \{j_l \in \mathbf{N}\}_{l=2}^m$ *such that* $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ *and for every* $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ *for each* $k \in \mathbf{N} \backslash \{i_l, j_l\}$ *and* $\mathbf{z}^l > \mathbf{x}^0$ *implies* $u_{i_l}(z_{i_l}^{l-1}; z_{i_l}^l) u_{j_l}(z_{j_l}^{l-1}; z_{j_l}^l) \geq 1$. *Moreover, if* $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} \leq 2$, *then this condition holds for* $m = 2$
(2) $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} \leq 2$ *and not* $\mathbf{y} > \mathbf{x}^0$ *implies* $\mathbf{x} \succsim \mathbf{y}$
(3) $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} \leq 2$ *and not* $\mathbf{x} > \mathbf{x}^0$ *and* $\mathbf{x} \succsim \mathbf{y}$ *implies not* $\mathbf{y} > \mathbf{x}^0$.

In order to analyze the social welfare function with respect to the axioms that characterize it, we first investigate sets of profiles for which the *ON* social welfare function satisfies the axioms. Lemma 4.1 is used in the following proposition, which is an analogue of proposition 3.2. The proposition presents necessary and sufficient conditions on the domain of preference profiles for the *ON* social preference to be a transitive binary relation on the set $\mathbf{X}$ (proof in the appendix).

PROPOSITION 4.1. *Let* $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{ms})^n$ *and* $\succsim = ON(\langle \succsim_k \rangle_{k=1}^n)$. *Then* $\succsim$ *is transitive if, and only if,* $\succsim_k$ $(k \in \mathbf{N})$ *have multiplicatively separable representations* $g_k v_k$, *such that* $g_i = g_j$, $\forall i, j \in \mathbf{N}$.

Proposition 4.1 suggests that the **UD** property and the definitions of the sets $\mathcal{D}^{UD}(g)$ and $\mathcal{D}^{UD}$ apply also in the case where $n > 2$. That is, the social preference is transitive only for profiles in $\mathcal{D}^{UD}$. The conditions for transitivity are again stated on the probability distortion functions g_k alone due to the multiplicatively separable property of the preference relations in $\mathcal{P}^{ms}$.

Proposition 4.2 and its interpretations are analogous to part (1) of theorem 3.1. The proposition states conditions under which the function $ON(\cdot)$ is well defined and has a Nash-like utility product representation (proof in the appendix). Again, Let $D^C \subseteq (\mathcal{P}^{ms})^n$ be the set of all $\langle \succsim_k \rangle_{k=1}^n$ for which $\succsim = ON(\langle \succsim_k \rangle_{k=1}^n)$ is a complete binary relation (the analysis of the set D^C is postponed to a later stage).

PROPOSITION 4.2. *Let* $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}^C \cap \mathcal{D}^{UD}$, *where* $\succsim_k$ *have multiplicatively separable representations* $g v_k$, *and let* $\succsim = ON(\langle \succsim_k \rangle_{k=1}^n)$. *Then, for any* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \prod_k v_k(x_k) \geq \prod_k v_k(y_k)$.

Note that the function $ON(\cdot)$ is well defined on the domain $\mathcal{D}^C \cap \mathcal{D}^{UD}$, i.e. it yields a complete and transitive binary relation. Transitivity and completeness of the social preference follows from the corresponding properties of the relation $\geq$ on the set $\{\prod_k v_k(x_k) \in \mathbb{R}_+ | \; \mathbf{x} \in \mathbf{X}\}$.

The axioms **PAR** and **ANM** do not impose any further constraints on the domain investigated. Unlike the case of $n = 2$, a different conclusion holds for the axiom **IIA** since there exist domains for which the *ON* social welfare function does not satisfy this axiom. Proposition 4.3 presents necessary and sufficient conditions on domains of the function $ON(\cdot)$ for which it is well defined, under which the function also satisfies the **IIA** axiom. Note that we are still not confined to the expected utility case.

PROPOSITION 4.3. *Let* $\mathcal{D} \subseteq \mathcal{D}^C \cap \mathcal{D}^{UD}$. *Then* $ON(\cdot)$ *satisfies* ***IIA*** *on* $\mathcal{D}$ *if and only if* $\mathcal{D} \subseteq \mathcal{D}^C \cap \mathcal{D}^{UD}(g)$ *for some* $g \in \mathcal{G}$.

Proof. Let $\langle\succsim_k\rangle_{k=1}^n, \langle\tilde{\succsim}_k\rangle_{k=1}^n \in \mathcal{D}$ and $g, \tilde{g} \in \mathcal{G}$ such that $\langle\succsim_k\rangle_{k=1}^n \in \mathcal{D}^{UD}(g)$, $\langle\tilde{\succsim}_k\rangle_{k=1}^n \in \mathcal{D}^{UD}(\tilde{g})$ and $\succsim_k, \tilde{\succsim}_k$ are represented for any elementary lottery $p\mathbf{x}$ by $g(p)v_k(x_k)$ and $\tilde{g}(p)\tilde{v}_k(x_k)$, respectively. Let $\succsim = ON(\langle\succsim_k\rangle_{k=1}^n)$, $\tilde{\succsim} = ON(\langle\tilde{\succsim}_k\rangle_{k=1}^n)$ and $u_k, \tilde{u}_k$ be the induced utility functions of $\succsim_k, \tilde{\succsim}_k$, respectively. Suppose $\mathcal{D} \subseteq \mathcal{D}^C \cap \mathcal{D}^{UD}(g)$, thus $g = \tilde{g}$. Verifying IIA, suppose $\mathbf{x}\succsim\mathbf{y} > \mathbf{x}^0$, then $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$ by proposition 4.2 and for every $k \in \mathbf{N}$, $u_k(x_k; y_k) = \tilde{u}_k(\tilde{x}_k; \tilde{y}_k)$ implies $\frac{v_k(x_k)}{v_k(y_k)} = \frac{\tilde{v}_k(\tilde{x}_k)}{\tilde{v}_k(\tilde{y}_k)}$, thus $\prod_k \tilde{v}_k(\tilde{x}_k) \geq \prod_k \tilde{v}_k(\tilde{y}_k)$, therefore $\tilde{\mathbf{x}}\tilde{\succsim}\tilde{\mathbf{y}}$. If not $\mathbf{y} > \mathbf{x}^0$ then $\prod_k v_k(y_k) = \prod_k \tilde{v}_k(\tilde{y}_k) = 0 \leq \prod_k \tilde{v}_k(\tilde{x}_k)$, thus $\tilde{\mathbf{x}}\tilde{\succsim}\tilde{\mathbf{y}}$ by proposition 4.2. Hence, $ON(\cdot)$ satisfies **IIA** on $\mathcal{D}$.
To prove the converse, suppose $ON(\cdot)$ satisfies **IIA** on $\mathcal{D} \subseteq \mathcal{D}^{UD} \cap \mathcal{D}^{SSLC}$, and let $\alpha, \beta \in [0,1]$. Then, by continuity and monotonicity of $v_k(\cdot)$, $v_k(x_k^0) = 0$ and $\mathbf{X}$ being $\mathbf{x}^0$-comprehensive, there exist $\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ and $i, j, l \in \mathbf{N}$ such that $x_k = y_k$ and $\tilde{x}_k = \tilde{y}_k$ for every $k \in \mathbf{N}\backslash\{i, j, l\}$, $\frac{v_i(x_i)}{v_i(y_i)} = \alpha$, $\frac{v_j(x_j)}{v_j(y_j)} = \beta$, $\frac{v_l(y_l)}{v_l(x_l)} = \alpha\beta$, $\frac{\tilde{v}_i(\tilde{x}_i)}{\tilde{v}_i(\tilde{y}_i)} = \tilde{g}[g^{-1}(\alpha)]$, $\frac{\tilde{v}_j(\tilde{x}_j)}{\tilde{v}_j(\tilde{y}_j)} = \tilde{g}[g^{-1}(\beta)]$ and $\frac{\tilde{v}_l(\tilde{y}_l)}{\tilde{v}_l(\tilde{x}_l)} = \tilde{g}[g^{-1}(\alpha\beta)]$. Thus, $\prod_k v_k(x_k) = \prod_k v_k(y_k)$, therefore $\mathbf{x} \sim \mathbf{y}$ by proposition 4.2. Furthermore, for every $k \in \mathbf{N}$, $\tilde{u}_k(\tilde{x}_k; \tilde{y}_k) = u_k(x_k; y_k)$, thus $\tilde{\mathbf{x}}\tilde{\sim}\tilde{\mathbf{y}}$ by **IIA**, therefore $\prod_k \tilde{v}_k(\tilde{x}_k) = \prod_k \tilde{v}_k(\tilde{y}_k)$ by proposition 4.2. It follows that for every $\alpha, \beta \in [0,1]$,

$$\tilde{g}[g^{-1}(\alpha\beta)] = \tilde{g}[g^{-1}(\alpha)] \cdot \tilde{g}[g^{-1}(\beta)].$$

Fix $\alpha \in (0,1)$. Then, for every $l, m \geq 1$, $\tilde{g}[g^{-1}(\alpha^{\frac{l}{m}})] = \tilde{g}[g^{-1}(\alpha^{\frac{l-1}{m}})] \cdot \tilde{g}[g^{-1}(\alpha^{\frac{1}{m}})]$, thus by induction $\tilde{g}[g^{-1}(\alpha^{\frac{l}{m}})] = [\tilde{g}[g^{-1}(\alpha^{\frac{1}{m}})]]^l$. For $l = m$, $\tilde{g}[g^{-1}(\alpha)] = [\tilde{g}[g^{-1}(\alpha^{\frac{1}{m}})]]^m$, thus for every $l, m \geq 1$, $\tilde{g}[g^{-1}(\alpha^{\frac{l}{m}})] = [\tilde{g}[g^{-1}(\alpha)]]^{\frac{l}{m}}$. Therefore, by continuity of $\tilde{g}[g^{-1}(\cdot)]$, for every $\gamma > 0$, $\tilde{g}[g^{-1}(\alpha^\gamma)] = [\tilde{g}[g^{-1}(\alpha)]]^\gamma$. It follows that for $\theta(\alpha) = \frac{\log \tilde{g}[g^{-1}(\alpha)]}{\log \alpha} > 0$ and every $\beta \in [0,1]$, $\tilde{g}[g^{-1}(\beta)] = \beta^{\theta(\alpha)}$. Furthermore, for any $\alpha' \in (0,1)$ and $\beta \in [0,1]$, $\tilde{g}[g^{-1}(\beta)] = \beta^{\theta(\alpha')}$, thus $\theta(\alpha) = \theta(\alpha')$. Therefore, for any $\alpha \in (0,1)$, $\theta(\alpha)$ is constant and its value is denoted θ. Hence, for every $\beta \in [0,1]$, $\tilde{g}[g^{-1}(\beta)] = \beta^\theta$. It follows that for every $p \in [0,1]$, $\tilde{g}(p) = [g(p)]^\theta$. Thus, for every $k \in \mathbf{N}$, $\tilde{\succsim}_k$ is represented for any elementary lottery $p\mathbf{x}$ by $g(p)[\tilde{v}_k(x_k)]^{1/\theta}$. Hence $\mathcal{D} \subseteq \mathcal{D}^{UD}(g)$. ∎

In order to ensure completeness, it is possible to add a smoothness and convexity assumption as sufficient conditions. Here, as a result of the separability property, the extra assumption may be applied only to the value functions v_k and the set $\mathbf{X}$, not to the distortion functions g_k. Let $\mathcal{P}^{Dms} \subseteq \mathcal{P}^{ms}$ be the set of *smooth* multiplicatively separable preference relations, in other words preference relations for which the function v in the representation is also differentiable (the function g in the representation need not be differentiable). We also take a further assumption on the set $\mathbf{X}$ to have a smooth boundary $\partial\mathbf{X}$ at all $\mathbf{x} \in \partial\mathbf{X}$ for which $\prod_k (x_k - x_k^0) > 0$. The convexity condition we suggest is not very strong since it merely applies to the set of log-values of the social states in $\mathbf{X}$.

Definition 4.3. **SSLC** (Smoothness and Strict Log-Convexity): A preference relation profile $\langle \succsim_k \rangle_{k=1}^n \in (\mathcal{P}^{Dms})^n$ satisfies **SSLC** with respect to the alternative set $\mathbf{X}$, if there exists a smooth multiplicatively separable representation, $g_k v_k$, such that $\{\langle \log v_k(x_k) \rangle_{k=1}^n \in \mathbb{R}^n \mid \mathbf{x}^0 < \mathbf{x} \in \mathbf{X}\}$ is a strictly convex set.[4]

Definition 4.4. Let $\mathcal{D}^{SSLC} \subseteq (\mathcal{P}^{Dms})^n$ be the set of all profiles $\langle \succsim_k \rangle_{k=1}^n$ that satisfy **SSLC** with respect to the alternative set $\mathbf{X}$.

Note that in order to check whether a preference profile satisfies the property **SSLC**, any combination of the possible multiplicatively separable representations $g_k v_k$ may be chosen for the individual preferences in that profile. This is true because any multiplicatively separable representation of $\succsim_k$ is of the form $(g_k)^{\alpha_k}(v_k)^{\alpha_k}$, for some $\alpha_k > 0$. Thus, checking whether the set $\{\langle \log[v_k(x_k)]^{\alpha_k} \rangle_{k=1}^n \in \mathbb{R}^n \mid \mathbf{x}^0 < \mathbf{x} \in \mathbf{X}\}$ is strictly convex does not depend on the value of α_k chosen. Note also, that the set $\mathcal{D}^{SSLC}$ depends on the geometric shape of the set $\mathbf{X}$. Thus, if $\mathbf{X}$ is not symmetric, given a profile in $\mathcal{D}^{SSLC}$, not all the permutations of that profile are necessarily in $\mathcal{D}^{SSLC}$. However, since $\mathbf{X}$ has a smooth boundary, there always exist profiles in $\mathcal{D}^{SSLC}$ for which the value functions v_k are differentiable and sufficiently locally concave, such that all the permutations of these profiles are indeed in $\mathcal{D}^{SSLC}$.

The **SSLC** property implies geometric conditions which are sufficient for the completeness of the ON social preference relation. These conditions are presented in lemma 5.1 and are used in the following proposition 4.4 (the lemma is presented in the appendix). Proposition 4.4 is analogous to corollary 3.1. The difference between the two is the addition of the **SSLC** condition here to ensure completeness of the social preference in the case where $n > 2$. The proposition states conditions under which the function $ON(\cdot)$ yields a complete binary relation.

Proposition 4.4. *The domain $\mathcal{D}^{SSLC} \cap \mathcal{D}^{UD} \subseteq \mathcal{D}^C$, i.e. the function $ON(\cdot)$ yields a complete binary relation on $\mathcal{D}^{SSLC} \cap \mathcal{D}^{UD}$.*

[4] A set $A \subseteq \mathbb{R}^n$ is strictly convex if for any $\mathbf{a}, \mathbf{b} \in A$ and $\lambda \in (0,1)$, $\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ is an interior point of A.

Proof. By completeness of the relation $\geq$ on the set $\{\prod_k v_k(x_k) \in \mathbb{R}_+ | \mathbf{x} \in \mathbf{X}\}$, it suffices to show that for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$ implies $\mathbf{x} \succsim \mathbf{y}$. Let $u_k(\cdot;\cdot)$ represent the induced utility functions of $\succsim_k$. Suppose $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$ and assume first that $\prod_k v_k(x_k), \prod_k v_k(y_k) > 0$. Let $C = \{\langle v_k(z_k)\rangle_{k=1}^n \in \mathbb{R}_+^n \mid \mathbf{x^0} < \mathbf{z} \in \mathbf{X}\}$ and $\mathbf{a}, \mathbf{b} \in C$ correspond to $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. The conditions of lemma 5.1 hold since $\mathbf{X}$ is compact, $\mathbf{x}^0$-comprehensive, has a smooth boundary $\partial\mathbf{X}$ at all $\mathbf{x} \in \partial\mathbf{X}$ for which $\prod_k (x_k - x_k^0) > 0$, $v_k(x_k^0) = 0$, the set $\{\langle \log c_k\rangle_{k=1}^n \mid \mathbf{0} < \mathbf{c} \in C\}$ is strictly convex, v_k are differentiable and $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$. Thus, there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{\mathbf{c}^l \in C\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{c}^l = \langle v_k(z_k^l)\rangle_{k=1}^n$, $c_k^{l-1} = c_k^l$ for each $k \in \mathbf{N}\backslash\{i_l, j_l\}$ and $c_{i_l}^{l-1} c_{j_l}^{l-1} \geq c_{i_l}^l c_{j_l}^l$. Moreover, if $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} \leq 2$, then this condition holds for $m = 2$. If $\mathbf{c}^{l-1}, \mathbf{c}^l > \mathbf{0}$ then $\mathbf{z}^{l-1}, \mathbf{z}^l > \mathbf{x}^0$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N}\backslash\{i_l, j_l\}$ and either $c_k^{l-1} \geq c_k^l$ for $k \in \{i_l, j_l\}$ or $u_{i_l}(z_{i_l}^l; z_{i_l}^{l-1}) = g^{-1}(\frac{c_{i_l}^l}{c_{i_l}^{l-1}}) \leq g^{-1}(\frac{c_{j_l}^{l-1}}{c_{j_l}^l}) = u_{j_l}(z_{j_l}^{l-1}; z_{j_l}^l)$, thus $u_{i_l}(z_{i_l}^l; z_{i_l}^{l-1}) u_{j_l}(z_{j_l}^l; z_{j_l}^{l-1}) \geq 1$. If not $\mathbf{c}^l > 0$ then not $\mathbf{z}^l > \mathbf{x}^0$. Hence, $\mathbf{x} \succsim \mathbf{y}$ according to lemma 4.1. Suppose that $\prod_k v_k(y_k) = 0$. If $\prod_k v_k(x_k) > 0$ then there exist $\hat{\mathbf{y}} \in \mathbf{X}$ for which $\#\{k \in \mathbf{N} \mid \hat{y}_k \neq y_k\} \leq 2$ and $\prod_k v_k(x_k) \geq \prod_k v_k(\hat{y}_k) > \prod_k v_k(y_k)$, thus $\mathbf{x} \succsim \hat{\mathbf{y}} \succsim \mathbf{y}$ by lemma 4.1 and therefore $\mathbf{x} \succsim \mathbf{y}$ by transitivity of $\succsim$. If $\prod_k v_k(x_k) = \prod_k v_k(y_k) = 0$, then there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N}\backslash\{i_l, j_l\}$ and $\prod_k v_k(z_k^l) = 0$. Thus $\mathbf{z}^{l-1} \succsim \mathbf{z}^l$ by lemma 4.1 and therefore $\mathbf{x} \succsim \mathbf{y}$. ▮

As a corollary of propositions 4.2 and 4.4, in the special case of DL and EU preferences, where $g(p) = p$, $\forall p \in [0,1]$ and v_k represent the value functions corresponding to $\succsim_k$, we achieve sufficient conditions for the function $ON(\cdot)$ to be well defined and the social preference to be represented by the individual value product. When $n > 2$, these sufficient conditions are log-concavity and differentiability of the functions v_k, assuming that set $\mathbf{X}$ is convex and has a smooth efficient boundary (when $n = 2$ these conditions are unnecessary as shown by theorem 3.1).

Example 4.2. Considering again example 4.1, the conditions of proposition 4.4 hold. We can conclude, according to the ON social preference, that $\mathbf{x} = (3,3,3) \succ \mathbf{y} = (1,4,4)$ since the utility product 27 is strictly larger than the utility product 16. Furthermore, $\mathbf{x}$ is the unique optimal social state in $\mathbf{X}$, since it is the only state that maximizes the individual utility product gain over the social state $\mathbf{x}^0$.

5. CHARACTERIZATION OF THE ORDINAL NASH SOCIAL WELFARE FUNCTION

We now provide a characterization of the ON social welfare function over domains $\mathcal{D}$ over which it is well defined. Theorem 5.1 characterizes the social welfare function on the domains investigated above. In the case where $n = 2$, the axiomatization is carried out for the set $\mathcal{D}^{UD}$, the maximal domain contained in $(\mathcal{P}^{ms})^n$ for which the function is well defined, as stated in theorem 3.1. In the case where $n > 2$, the axiomatization is restricted to domains that contain the set $\mathcal{D}^{UD}(g) \cap \mathcal{D}^{SSLC}$ for some $g \in \mathcal{G}$, as required by propositions 4.3 and 4.4.

THEOREM 5.1. *Let $W(\cdot)$ satisfy **MS**, **WO**, **PAR**, **ANM** and **IIA** on $\mathcal{D} \subseteq \mathcal{D}^C$. Then:*
(1) $W = ON$
(2) If $n = 2$ then $\mathcal{D} \subseteq \mathcal{D}^{UD}$, e.g $\mathcal{D} = \mathcal{D}^{UD}$
(3) If $n > 2$ then for some $g \in \mathcal{G}$, $\mathcal{D} \subseteq \mathcal{D}^{UD}(g)$, e.g $\mathcal{D} = \mathcal{D}^{UD}(g) \cap \mathcal{D}^{SSLC}$
(4) For every $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$ there exist a multiplicatively separable representation $\langle gv_k \rangle_{k=1}^n$ such that for every $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$.

Proof. The case of $n = 2$ is shown by theorem 3.1. We provide a proof for the case of $n > 2$. Let A_W be the set defined in proposition 2.1 and let $\mathbf{s} \in \mathbb{R}^n_+$. Suppose first that there exist $i, j \in \mathbf{N}$ for which $s_k = 1$ for each $k \in \mathbf{N} \backslash \{i, j\}$. Then $\mathbf{s} \in A_W$ if, and only if, $s_i s_j \geq 1$. Furthermore, for any $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that there exist $i, j \in \mathbf{N}$ for which $\mathbf{x} \sim_k \mathbf{y}$ for each $k \in \mathbf{N} \backslash \{i, j\}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\mathbf{y} > \mathbf{x}^0$ implies $u_i(x_i; y_i) u_j(x_j; y_j) \geq 1$. The proof is similar to the one given for proposition 3.1 applied for individuals i, j instead of $1, 2$, two profiles for which only i, j are permuted, vectors $\mathbf{t}, \tilde{\mathbf{s}}$ for which all components except i, j equal to 1 and states $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for which $\mathbf{x} \sim_k \mathbf{y} \sim_k \mathbf{z}$ for each $k \in \mathbf{N} \backslash \{i, j\}$. Hence, $\mathcal{D} \subseteq \mathcal{D}^{UD}$ by lemma 4.1 and proposition 4.1.
Let $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$, any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and let gv_k be multiplicatively separable representations of $\succsim_k$. Suppose that $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$, then by theorem 4.2 $\mathbf{x} \breve{\succsim} \mathbf{y}$, where $\breve{\succsim} = ON(\langle \succsim_k \rangle_{k=1}^n)$. Thus by lemma 4.1 there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \backslash \{i_l, j_l\}$ and $\mathbf{z}^l > \mathbf{x}^0$ implies $u_{i_l}(z_{i_l}^{l-1}; z_{i_l}^l) u_{j_l}(z_{j_l}^{l-1}; z_{j_l}^l) \geq 1$. Therefore for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \succsim \mathbf{z}^l$, which implies $\mathbf{x} \succsim \mathbf{y}$ by transitivity. Suppose now that $\prod_k v_k(x_k) < \prod_k v_k(y_k)$ and $\mathbf{y} \breve{\succ} \mathbf{x}$. Thus, there exist sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{y}$, $\mathbf{z}^m = \mathbf{x}$ and there exist $2 \leq \hat{l} \leq m$ such that $\mathbf{z}^l > \mathbf{x}^0$ implies $u_{i_{\hat{l}}}(z_{i_{\hat{l}}}^{\hat{l}-1}; z_{i_{\hat{l}}}^{\hat{l}}) u_{j_{\hat{l}}}(z_{j_{\hat{l}}}^{\hat{l}-1}; z_{j_{\hat{l}}}^{\hat{l}}) > 1$. Therefore $\mathbf{z}^{\hat{l}-1} \succ \mathbf{z}^{\hat{l}}$, which implies $\mathbf{y} \succ \mathbf{x}$ by

transitivity. Hence, for any $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$, where gv_k are multiplicatively separable representations of $\succsim_k$. Hence, $W(\langle \succsim_k \rangle_{k=1}^n) = ON(\langle \succsim_k \rangle_{k=1}^n)$ by theorem 4.2, proving (1) and (4). Part (3) of the theorem as well as $ON(\cdot)$ satisfying the axioms follows immediately from propositions 4.1, 4.3 and 4.4 and the claims given for **PAR** and **ANM** in the proof of theorem 3.1. ▌

We can now provide a full characterization of the set A_W defined in proposition 2.1. Since the only information relevant to the social aggregation is included in the state value functions, we need a function that takes from the induced utilities only the relevant part. This function depends on the function g which is uniform on the domain chosen. Given $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}^{UD}(g)$, let the function $h_g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined such that

$$h_g(r) = \begin{cases} g(r) & \text{if } r \leq 1 \\ [g(r^{-1})]^{-1} & \text{if } r > 1 \end{cases}.$$

Corollary 5.1. *Let $W(\cdot)$ satisfy **MS**, **WO**, **PAR**, **ANM** and **IIA** on $\mathcal{D} \subseteq \mathcal{D}^C$ and let A_W be the corresponding set. Then for any $\mathbf{s} \in \mathbb{R}_+^n$, $\mathbf{s} \in A_W$ if, and only if, $\prod_k h_g(s_k) \geq 1$.*

The condition given in corollary 5.1 (proof in the appendix) coincides with the requirement $s_1 s_2 \geq 1$ that characterized the set A_W in the case where only 2 individuals care about the choice.

Appendix

Proof. Lemma 4.1. By definition, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if, and only if, there exist $m \geq 2$ and sequences $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m, \{i_l \in \mathbf{N}\}_{l=2}^m, \{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{z}^1 = \mathbf{x}$, $\mathbf{z}^m = \mathbf{y}$ and for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N} \backslash \{i_l, j_l\}$ and either $\mathbf{z}^{l-1} \succsim_k \mathbf{z}^l$ for $k \in \{i_l, j_l\}$ or there exist $p_l, q_l \in [0,1]$ such that $p_l \mathbf{z}^{l-1} \sim_{i_l} \mathbf{z}^l$, $q_l \mathbf{z}^l \sim_{j_l} \mathbf{z}^{l-1}$ and $p_l \leq q_l$. Moreover, if $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} \leq 2$, then this condition holds for $m = 2$. For any $\mathbf{w}, \hat{\mathbf{w}} \in \mathbf{X}$ and $i, j \in \mathbf{N}$ such that $\mathbf{w} \sim_k \hat{\mathbf{w}}$ for each $k \in \mathbf{N} \backslash \{i, j\}$, either $\mathbf{w} \succsim_k \hat{\mathbf{w}}$ for both i, j or there exist $p, q \in [0,1]$ such that $p\mathbf{w} \sim_i \hat{\mathbf{w}}$, $q\hat{\mathbf{w}} \sim_j \mathbf{w}$, where by definition $p = u_i(\hat{w}_i; w_i)$, $q = u_j(w_j; \hat{w}_j)$. Therefore, if $\hat{\mathbf{w}} > \mathbf{x}^0$ then $u_i(w_i; \hat{w}_i) u_j(w_j; \hat{w}_j) \geq 1$ if, and only if, either $\mathbf{w} \succsim_k \hat{\mathbf{w}}$ for both i, j or $p \leq q$. If not $\hat{\mathbf{w}} > \mathbf{x}^0$ then either $\mathbf{w} \succsim_k \hat{\mathbf{w}}$ for both i, j or $0 = p \leq q$. Hence, condition (1)(a) holds if, and only if, condition (1)(b) holds. Suppose that $\#\{k \in \mathbf{N} \mid x_k \neq y_k\} \leq 2$, then (2) follows immediately and (3) also holds since $\mathbf{y} > \mathbf{x}^0$ implies $\prod_k u_k(x_k; y_k) = 0$ and thus not $\mathbf{x} \succsim \mathbf{y}$, a contradiction. ▌

Proof. Proposition 4.1. Let $u_k(\cdot; \cdot)$ be the induced utility functions of $\succsim_k$. Suppose there exist functions $g, v_1, ..., v_n$ such that for any $k \in \mathbf{N}$, $\succsim_k$

can be represented for any $p\mathbf{x} \in \mathcal{E}$ by $g(p)v_k(x_k)$, where $v_k(x_k^0) = 0$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that $\mathbf{x} \succsim \mathbf{y} \succsim \mathbf{z}$. Then, there exist $l_1, l_2 \geq 1$ such that $\mathbf{x} \succsim^{l_1} \mathbf{y} \succsim^{l_2} \mathbf{z}$, which implies $\mathbf{x} \succsim^{m} \mathbf{z}$, where $m = l_1 + l_2$ and $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$ and $\{j_l \in \mathbf{N}\}_{l=2}^m$ are the corresponding sequences. If $\#\{k \in \mathbf{N} \mid x_k \neq z_k\} > 2$ or $\mathbf{x} \geq \mathbf{z}$ or ($\#\{k \in \mathbf{N} \mid x_k \neq z_k\} \leq 2$ and not $\mathbf{z} > \mathbf{x}^0$) then $\mathbf{x} \succsim \mathbf{z}$ by lemma 4.1. Otherwise, for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N}\backslash\{i_l, j_l\}$. Moreover, if not $\mathbf{z}^l > \mathbf{x}^0$ then $\prod_k v_k(z_k^{l-1}) \geq \prod_k v_k(z_k^l) = 0$ and if $\mathbf{z}^{l-1}, \mathbf{z}^l > \mathbf{x}^0$ then either $u_k(z_k^{l-1}; z_k^l) \geq 1$ for $k \in \{i_l, j_l\}$ or

$$g^{-1}(\frac{v_{i_l}(z_{i_l}^l)}{v_{i_l}(z_{i_l}^{l-1})}) = u_{i_l}(z_{i_l}^l; z_{i_l}^{l-1}) \leq u_{j_l}(z_{j_l}^{l-1}; z_{j_l}^l) = g^{-1}(\frac{v_{j_l}(z_{j_l}^{l-1})}{v_{j_l}(z_{j_l}^l)}).$$

Therefore, $\prod_k v_k(z_k^{l-1}) \geq \prod_k v_k(z_k^l)$ and thus $\prod_k v_k(x_k) \geq v_k(z_k)$. Let $\hat{\imath}, \hat{\jmath} \in \mathbf{N}$ such that $\mathbf{z} \succsim_{\hat{\imath}} \mathbf{x}$ and $\mathbf{x} \succsim_{\hat{\jmath}} \mathbf{z}$ and $\mathbf{x} \sim_k \mathbf{z}$ for each $k \in \mathbf{N}\backslash\{\hat{\imath}, \hat{\jmath}\}$. Then, $\frac{v_{\hat{\imath}}(x_{\hat{\imath}})}{v_{\hat{\imath}}(z_{\hat{\imath}})} \frac{v_{\hat{\jmath}}(x_{\hat{\jmath}})}{v_{\hat{\jmath}}(z_{\hat{\jmath}})} \geq 1$ and therefore $u_{\hat{\imath}}(x_{\hat{\imath}}; z_{\hat{\imath}}) = g^{-1}(\frac{v_{\hat{\imath}}(x_{\hat{\imath}})}{v_{\hat{\imath}}(z_{\hat{\imath}})}) \geq g^{-1}(\frac{v_{\hat{\jmath}}(z_{\hat{\jmath}})}{v_{\hat{\jmath}}(x_{\hat{\jmath}})}) = u_{\hat{\jmath}}(z_{\hat{\jmath}}; x_{\hat{\jmath}})$. Thus $\mathbf{x} \succsim^1 \mathbf{z}$ by lemma 4.1 and therefore $\mathbf{x} \succsim \mathbf{z}$. Hence $\succsim$ is transitive.

The proof of the converse is the same as the corresponding part of the proof of proposition 3.2, applied for every $k \in \mathbf{N}\backslash\{1\}$, with the addition that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ are chosen in the proof such that $x_i = y_i = z_i$ for every $i \in \mathbf{N}\backslash\{k, 1\}$. ▌

Proof. Proposition 4.2. Let $u_k(\cdot;\cdot)$ represent the induced utility functions of $\succsim_k$. Suppose $\mathbf{x} \succsim \mathbf{y}$. By lemma 4.1, there exist $\{\mathbf{z}^l \in \mathbf{X}\}_{l=1}^m, \{i_l \in \mathbf{N}\}_{l=2}^m, \{j_l \in \mathbf{N}\}_{l=2}^m$ such that for every $2 \leq l \leq m$, $\mathbf{z}^{l-1} \sim_k \mathbf{z}^l$ for each $k \in \mathbf{N}\backslash\{i_l, j_l\}$. Moreover, if not $\mathbf{z}^l > \mathbf{x}^0$ then $\prod_k v_k(z_k^{l-1}) \geq \prod_k v_k(z_k^l) = 0$ and if $\mathbf{z}^l > \mathbf{x}^0$ then either $u_k(z_k^{l-1}; z_k^l) \geq 1$ for $k \in \{i_l, j_l\}$ or $g^{-1}(\frac{v_{i_l}(z_{i_l}^l)}{v_{i_l}(z_{i_l}^{l-1})}) \leq g^{-1}(\frac{v_{j_l}(z_{j_l}^{l-1})}{v_{j_l}(z_{j_l}^l)})$. Therefore, $\prod_k v_k(z_k^{l-1}) \geq \prod_k v_k(z_k^l)$ and thus $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$. To prove the converse, suppose $\mathbf{y} \succ \mathbf{x}$. Then there exist $2 \leq \hat{l} \leq m$ such that $v_k(z_k^{\hat{l}-1}) > \prod_k v_k(z_k^{\hat{l}})$ and thus $\prod_k v_k(y_k) > \prod_k v_k(x_k)$. Therefore $\prod_k v_k(x_k) \geq \prod_k v_k(y_k)$ implies $\mathbf{x} \succsim \mathbf{y}$ by completeness of $\succsim$. ▌

Proof. Corollary 5.1. Let $\mathbf{s} \in \mathbb{R}_+^n$. Assuming that $\mathbf{s}$ satisfies $s_k = 1$ for every $k \in \mathbf{N}\backslash\{i, j\}$, if $\mathbf{s} \geq \mathbf{1}$, then $\prod_k s_k \geq 1$ and $\prod_k h_g(s_k) \geq 1$. If not $\mathbf{s} \geq \mathbf{1}$, then we can assume without loss of generality that $s_i < 1$, thus $\prod_k s_k \geq 1$ if, and only if, $h_g(s_i) = g(s_i) \geq g(1/s_j) = 1/h_g(s_j)$ if, and only if, $\prod_k h_g(s_k) \geq 1$. Hence $\mathbf{s} \in A_W$ if, and only if, $\prod_k h_g(s_k) \geq 1$. Suppose that $\mathbf{s}$ satisfy $\#\{k \in \mathbf{N} \mid s_k \neq 1\} > 2$. Let $\langle \succsim_k \rangle_{k=1}^n \in \mathcal{D}$, $\succsim = W(\langle \succsim_k \rangle_{k=1}^n)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{y} > \mathbf{x}^0$ and $\mathbf{s} = \mathbf{u}_{\langle \succsim_k \rangle_{k=1}^n}(\mathbf{x}; \mathbf{y})$. For every $k \in \mathbf{N}$, $h_g(s_k) = h_g[u_k(x_k; y_k)] = \frac{v_k(x_k)}{v_k(y_k)}$, where gv_k are multiplicatively separable representations of $\succsim_k$. Then $\prod_k h_g(s_k) \geq 1 \iff \prod_k v_k(x_k) \geq \prod_k v_k(y_k)$ $\iff \mathbf{x} \succsim \mathbf{y} \iff \mathbf{s} \in A_W$. Hence $\mathbf{s} \in A_W$ if, and only if, $\prod_k h_g(s_k) \geq 1$. ∎

Lemma 5.1. *Let $C \subseteq \mathbb{R}^n_+$ be a compact and 0-comprehensive*[5] *set with smooth boundary ∂C at all $\mathbf{c} \in \partial C$ for which $\prod_k c_k > 0$, such that the set $\{\langle \log c_k \rangle_{k=1}^n \mid \mathbf{0} < \mathbf{c} \in \mathbf{C}\}$ is strictly convex. Let $\mathbf{a}, \mathbf{b} \in C$. Then, $\prod_k a_k \geq \prod_k b_k$ if and only if there exist $m \geq 2$ and sequences $\{\mathbf{c}^l \in C\}_{l=1}^m, \{i_l \in \mathbf{N}\}_{l=2}^m, \{j_l \in \mathbf{N}\}_{l=2}^m$ such that $\mathbf{c}^1 = \mathbf{a}$, $\mathbf{c}^m = \mathbf{b}$ and for every $2 \leq l \leq m$, $c_k^{l-1} = c_k^l$ for each $k \in \mathbf{N} \backslash \{i_l, j_l\}$ and $c_{i_l}^{l-1} c_{j_l}^{l-1} \geq c_{i_l}^l c_{j_l}^l$. Moreover, if $\#\{k \in \mathbf{N} \mid a_k \neq b_k\} \leq 2$, then this condition holds for $m = 2$. Furthermore, $\prod_k a_k > \prod_k b_k$ if and only if in addition there exist $2 \leq l \leq m$ such that $c_{i_l}^{l-1} c_{j_l}^{l-1} > c_{i_l}^l c_{j_l}^l$.*

Proof. Clearly, if there exist such sequences $\{\mathbf{c}^l \in C\}_{l=1}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=2}^m$, then for every $2 \leq l \leq m$, $\prod_k c_k^{l-1} \geq \prod_k c_k^l$, thus $\prod_k a_k \geq \prod_k b_k$. If in addition there exist $2 \leq h \leq m$ such that $c_{i_h}^{h-1} c_{j_h}^{h-1} > c_{i_h}^h c_{j_h}^h$ then clearly $\prod_k a_k > \prod_k b_k$.

To prove the converse, suppose $\prod_k a_k \geq \prod_k b_k$. Let $\mathbf{c}^1 = \mathbf{a}$ and let the sequences $\{\mathbf{c}^l \in C\}_{l=2}^m$, $\{i_l \in \mathbf{N}\}_{l=2}^m$, $\{j_l \in \mathbf{N}\}_{l=1}^m$ be constructed as follows, considering several cases.

(1) Suppose $\#\{k \in \mathbf{N} \mid a_k \neq b_k\} \leq 2$, then let $m = 2$, $\mathbf{c}^2 = \mathbf{b}$ and $i_2, j_2 \in \mathbf{N}$ such that $a_k = b_k$ for each $k \in \mathbf{N} \backslash \{i_2, j_2\}$, thus the required conditions are satisfied.

(2) Hereafter assume that $\#\{k \in \mathbf{N} \mid a_k \neq b_k\} > 2$. Suppose $\prod_k a_k > \prod_k b_k$. Since C is 0-comprehensive, there exist $\mathbf{c}^2 \in C$, $i_2, j_2 \in \mathbf{N}$ such that $c_k^2 = c_k^1$ for each $k \in \mathbf{N} \backslash \{i_2, j_2\}$, $c_{i_2}^2 < c_{i_2}^1$, $c_{j_2}^2 \leq c_{j_2}^1$ and $\prod_k c_k^2 = \prod_k b_k$, thus $c_{i_2}^1 c_{j_2}^1 > c_{i_2}^2 c_{j_2}^2$. The remaining sequences are constructed as in the case where $\prod_k a_k = \prod_k b_k$.

(3) Hereafter assume that $\prod_k a_k = \prod_k b_k$. Suppose $\prod_k a_k = 0$, then there exist $\hat{\imath}, \hat{\jmath} \in \mathbf{N}$ such that $a_{\hat{\imath}} = b_{\hat{\jmath}} = 0$. Let $m = \#\{k \in \mathbf{N} \backslash \{\hat{\jmath}\} \mid a_k \neq b_k\} + 1$ and for every $2 \leq l \leq m$, let $j_l = \hat{\jmath}$, $i_l \in \mathbf{N}$ such that $c_{i_l}^{l-1} \neq b_{i_l}$, and let $\mathbf{c}^l \in C$ such that $c_k^l = c_k^{l-1}$ for each $k \in \mathbf{N} \backslash \{i_l, j_l\}$, $c_{i_l}^l = b_{i_l}$ and $c_{j_l}^l = 0$. Clearly, these sequences satisfy the required conditions.

(4) Hereafter assume that $\prod_k a_k = \prod_k b_k > 0$. Suppose that $\mathbf{a}, \mathbf{b}$ are interior points of C. Let $\tilde{C} = \{\langle \log r_k \rangle_{k=1}^n \mid \mathbf{0} < \mathbf{r} \in C\}$. Note, that $\tilde{C}$ is a closed and strictly convex set. Let $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}^l$ be the interior points in $\tilde{C}$ corresponding to $\mathbf{a}, \mathbf{b}, \mathbf{c}^l$ in C. Then, there exist $0 < \delta < \left\| \tilde{\mathbf{b}} - \tilde{\mathbf{a}} \right\|$, such that for every $\tilde{\mathbf{r}} \in \mathbb{R}^n_+$ for which $\|\tilde{\mathbf{r}}\| < \delta$, $\tilde{\mathbf{a}} + \tilde{\mathbf{r}} \in \tilde{C}$ and $\tilde{\mathbf{b}} + \tilde{\mathbf{r}} \in \tilde{C}$. Let $m_1 = \left\lceil \left\| \tilde{\mathbf{b}} - \tilde{\mathbf{a}} \right\| / 2\delta \right\rceil$, $\varepsilon = \left\| \tilde{\mathbf{b}} - \tilde{\mathbf{a}} \right\| / 2m_1 \leq \delta$ and for every $0 \leq t \leq m_1$, let

[5] $\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n_+$, $\mathbf{b} \leq \mathbf{a} \in C \Rightarrow \mathbf{b} \in C$

$\tilde{\mathbf{s}}^t = \tilde{\mathbf{a}} + \frac{t}{m_1}(\tilde{\mathbf{b}} - \tilde{\mathbf{a}})$. By $\sum_k s_k^0 = \sum s_k^1$, $\tilde{\mathbf{s}}^0 \neq \tilde{\mathbf{s}}^1$ and $\mathbf{N}$ finite, there exist $m_2 > 2$ and extension of the sequences $i_l, j_l, \tilde{\mathbf{c}}^l$, as demonstrated in figure 2, such that for every $2 \leq l \leq m_2$, $\tilde{c}_{i_l}^{l-1} < \tilde{s}_{i_l}^1$, $\tilde{c}_{j_l}^{l-1} > \tilde{s}_{j_l}^1$, $\tilde{c}_k^l = \tilde{c}_k^{l-1}$ for each $k \in \mathbf{N} \backslash \{i_l, j_l\}$, $\alpha_l = \min\{\tilde{s}_{i_l}^1 - \tilde{c}_{i_l}^{l-1}, \tilde{c}_{j_l}^{l-1} - \tilde{s}_{j_l}^1\} > 0$, $\tilde{c}_{i_l}^l = \tilde{c}_{i_l}^{l-1} + \alpha_l$, $\tilde{c}_{j_l}^l = \tilde{c}_{j_l}^{l-1} - \alpha_l$ and $\tilde{\mathbf{c}}^{m_2} = \tilde{\mathbf{s}}^1$.

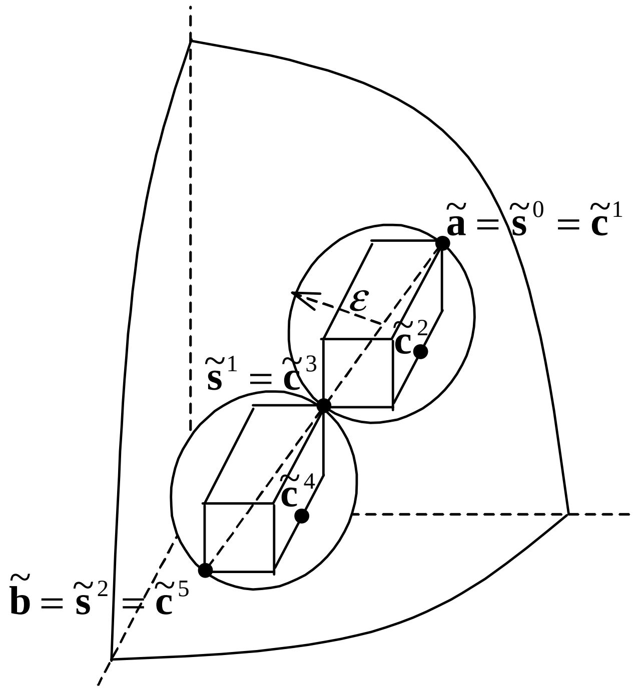

Figure 2: Construction of the sequences $\{\tilde{\mathbf{c}}^l \in \tilde{C}\}_{l=1}^5, \{i_l \in \mathbf{N}\}_{l=2}^5, \{j_l \in \mathbf{N}\}_{l=2}^5$

Then, for every $2 \leq l \leq m_2$ and $k \in \mathbf{N}$, $\min\{\tilde{s}_k^0, \tilde{s}_k^1\} \leq \tilde{c}_k^l \leq \max\{\tilde{s}_k^0, \tilde{s}_k^1\}$, thus

$$\left\|\tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^0 + \tilde{\mathbf{s}}^1)/2\right\| \leq \left\|\tilde{\mathbf{s}}^0 - \tilde{\mathbf{s}}^1\right\|/2 = \varepsilon.$$

Let $m_3 = m_1(m_2 - 1) + 1$ and extend the sequences $i_l, j_l, \tilde{\mathbf{c}}^l$ such that for every $2 \leq t \leq m_1, 2 \leq l \leq m_2$, let $i_{(t-1)(m_2-1)+l} = i_l$, $j_{(t-1)(m_2-1)+l} = j_l$ and $\tilde{\mathbf{c}}^{(t-1)(m_2-1)+l} = \tilde{\mathbf{c}}^l - \tilde{\mathbf{s}}^1 + \tilde{\mathbf{s}}^t = \tilde{\mathbf{c}}^l + \frac{t-1}{m_1}(\tilde{\mathbf{b}} - \tilde{\mathbf{a}})$, thus $\left\|\tilde{\mathbf{c}}^{(t-1)(m_2-1)+l} - (\tilde{\mathbf{s}}^{t-1} + \tilde{\mathbf{s}}^t)/2\right\| = \left\|\tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^0 + \tilde{\mathbf{s}}^1)/2\right\| \leq \varepsilon$. Then, $\tilde{\mathbf{c}}^{m_3} = \tilde{\mathbf{b}}$ and for every $2 \leq l \leq m_3$, there exist $1 \leq t_l \leq m_1$ such that $\left\|\tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^{t_l-1} + \tilde{\mathbf{s}}^{t_l})/2\right\| \leq \varepsilon$. Let $\tilde{\mathbf{r}}^l = \tilde{\mathbf{c}}^l - (\tilde{\mathbf{s}}^{t_l-1} + \tilde{\mathbf{s}}^{t_l})/2$. Then, $\tilde{\mathbf{c}}^l = \frac{2t_l-1}{2m_1}(\tilde{\mathbf{b}} + \tilde{\mathbf{r}}^l) + (1 - \frac{2t_l-1}{2m_1})(\tilde{\mathbf{a}} + \tilde{\mathbf{r}}^l) \in \tilde{C}$ by convexity of $\tilde{C}$. For every $2 \leq l \leq m_3$, let $\mathbf{c}^l \in C$ correspond to $\tilde{\mathbf{c}}^l$ in $\tilde{C}$, so that the required conditions are satisfied.

(5) Hereafter assume that either $\mathbf{a}$ or $\mathbf{b}$ are on the boundary ∂C of C. Suppose that $\mathbf{a}$ is on the boundary ∂C and $\mathbf{b}$ is an interior point of C. Construct $\mathbf{c}^2, i_2, j_2$ as follows: by strict convexity of $\tilde{C}$ and its smooth

boundary $\partial\tilde{C}$ at $\tilde{\mathbf{a}}$, there exists a unique (up to positive scalar multiplication) $\mathbf{t} \in \mathbb{R}^n_+$ such that for each $\tilde{\mathbf{r}} \in \tilde{C}\backslash\{\tilde{\mathbf{a}}\}$, $\sum_k t_k(\tilde{r}_k - \tilde{a}_k) < 0$. Suppose there exists $\lambda \neq 0$ such that for every $k \in \mathbf{N}$, $t_k = \lambda$. Since $\sum_k(\tilde{b}_k - \tilde{a}_k) = 0$, it follows that $\sum_k t_k(\tilde{b}_k - \tilde{a}_k) = 0$, and since $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$, then $\tilde{\mathbf{b}} \notin \tilde{C}$, a contradiction. Thus, there exist $i_2, j_2 \in \mathbf{N}$ such that $t_{i_2} < t_{j_2}$. Let $\tilde{C}_2 = \{\langle \tilde{r}_{i_2}, \tilde{r}_{j_2}\rangle \mid \tilde{\mathbf{r}} \in \tilde{C}, \forall k \in \mathbf{N}\backslash\{i_2, j_2\}, \tilde{r}_k = \tilde{a}_k\} \subseteq \mathbb{R}^2$, then $\tilde{C}_2$ is a closed and strictly convex set with smooth boundary $\partial\tilde{C}_2$ at $(\tilde{a}_{i_2}, \tilde{a}_{j_2})$. Therefore, (t_{i_2}, t_{j_2}) satisfies uniquely (up to positive scalar multiplication) for each $\tilde{\mathbf{r}} \in \tilde{C}_2\backslash\{(\tilde{a}_{i_2}, \tilde{a}_{j_2})\}$,

$$t_{i_2}(\tilde{r}_1 - \tilde{a}_{i_2}) + t_{j_2}(\tilde{r}_2 - \tilde{a}_{j_2}) < 0.$$

Since $(1,1)$ is not a scalar multiple of (t_{i_2}, t_{j_2}), the set $\tilde{D}_2 = \{\tilde{\mathbf{r}} \in \tilde{C}_2\backslash \{(\tilde{a}_{i_2}, \tilde{a}_{j_2})\} \mid (\tilde{r}_1 - \tilde{a}_{i_2}) + (\tilde{r}_2 - \tilde{a}_{j_2}) \geq 0\}$ is not empty. Moreover, there exist $\varepsilon > 0$ and $\tilde{\mathbf{d}} \equiv (\tilde{a}_{i_2} + \varepsilon, \tilde{a}_{j_2} - \varepsilon) \in \tilde{D}_2$, such that $\tilde{\mathbf{d}}$ is an interior point of $\tilde{C}_2$, as demonstrated in figure 3. Let $\tilde{\mathbf{c}}^2 \in \tilde{C}$ satisfy $\tilde{c}^2_k = \tilde{a}_k$ for each $k \in \mathbf{N}\backslash\{i_2, j_2\}$, $\tilde{c}^2_{i_2} = \tilde{d}_1$ and $\tilde{c}^2_{j_2} = \tilde{d}_2$.

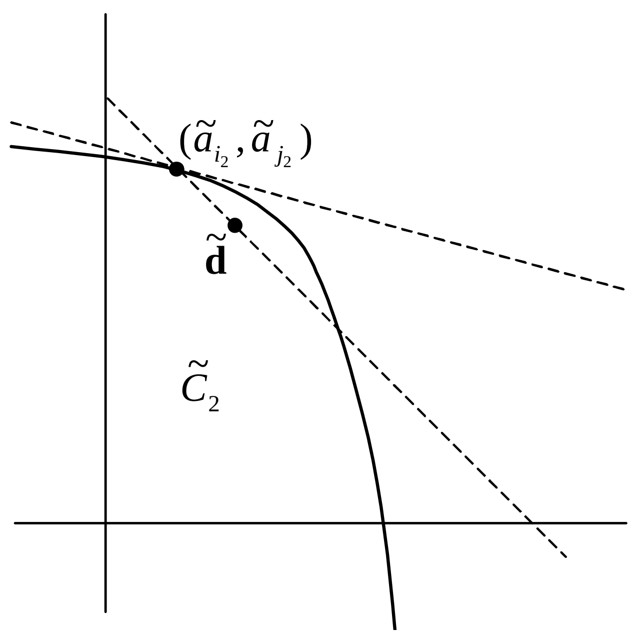

Figure 3: Existence of an interior point $\tilde{\mathbf{d}} \in \tilde{C}_2$

Let $\mathbf{c}^2 \in C$ be the corresponding point to $\tilde{\mathbf{c}}^2 \in \tilde{C}$. Clearly, $\mathbf{c}^2$ is an interior point of C.Note that $c^2_{i_2}c^2_{j_2} = c^1_{i_2}c^1_{j_2}$. The sequences $i_l, j_l, \mathbf{c}^l$ can then be extended as in case (4) to satisfy the required conditions.

(6) Assume now that $\mathbf{b}$ is on the boundary ∂C of C and $\mathbf{a}$ is an interior point of C. Similarly with respect to case (5), there exists $\hat{\mathbf{b}}$, an interior point of C, and $\hat{\imath}, \hat{\jmath} \in \mathbf{N}$ such that $\hat{b}_k = b_k$ for each $k \in \mathbf{N}\backslash\{\hat{\imath}, \hat{\jmath}\}$ and $\hat{b}_{\hat{\imath}}\hat{b}_{\hat{\jmath}} = b_{\hat{\imath}}b_{\hat{\jmath}}$.

Thus the sequences $i_l, j_l, \mathbf{c}^l$ will satisfy the required conditions when extended as in case (4) for $\mathbf{a}$ and $\hat{\mathbf{b}}$. ▌

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