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Manipulation through Bribes

by

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Abstract

We consider allocation rules that choose both a public outcome and transfers, based on the agents' reported valuations of the outcomes. Under a given allocation rule, a bribing situation exists when one agent could pay another to misreport his valuations, resulting in a net gain to both agents. A rule is *bribe-proof* if such opportunities never arise (including the case in which the briber and bribee are the same agent).

The central result is that under a *bribe-proof* rule, regardless of the domain of admissible valuations, the payoff to any one agent is a *continuous* function of any other agent's reported valuations. We then show that on connected domains of valuation functions, if either the set of outcomes is finite or each agent's set of admissible valuations is *smoothly* connected, then an agent's payoff is a *constant* function of other agents' reported valuations. Finally, under the additional assumption of a standard richness condition on the set of admissible valuations, a *bribe-proof* rule must be a constant function.

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1 Introduction

Consider the problem of deciding on a public outcome and on transfers of a private good among agents, based on how those agents value the various public outcomes. Without knowing the agents' valuations, it may not be reasonable for the decision maker to assume that they would truthfully report their valuations, if doing so is not in their best interest. For example, this is a concern in the literature on Clarke–Groves Mechanisms:[1] such decision rules are the only ones that are *efficient* (choose value-maximizing public outcomes) and are *strategy-proof* (never give any agent an incentive to misreport his valuations to the decision maker). Our concern here is to drop all distributional requirements, such as *efficiency*, and concentrate solely on incentives — we examine the consequences of adding an additional incentives requirement that applies to *pairs* of agents, as we define below.

One compelling, but strong, group incentive-compatibility requirement is *coalitional strategy-proofness*: no coalition of agents should be able to jointly misrepresent their valuations in a way that results in a direct gain to each of those agents. The desirability of this condition is clear. However, in many environments this condition is too strong, ruling out all by a few, unreasonable decision rules. Part of this strength comes from the fact that coalitions of *any* size are prevented from manipulating. However, the execution of a joint misrepresentation by a large number of agents requires that they all know each others' valuations and that they are able to coordinate their actions. In many situations, large coalitions may not be able to coordinate. On the other hand, it is reasonable to suspect that if a *small* coalition could manage to coordinate their actions in a profitable, joint misrepresentation of valuations, then they could, in addition, arrange transfers to each other, *i.e.* they could bribe each other to misrepresent.

We formulate the weakest intuitive condition that rules out this type of misrepresentation — one in which exactly *one* agent is bribed by one other agent to solely misrepresent his valuations. There are only two agents involved in the transfer, and only one of them misrepresents his type. Decision rules that eliminate the possibility of this type of behavior — and that are *strategy-proof* — are called *bribe-proof*.

[1] See Clarke (1971), Groves (1973), and Green and Laffont (1977).

The central result is that in a very general setting, if a solution is *bribe-proof*, then it satisfies a continuity property: The payoff an agent receives varies continuously with respect to changes in the reported valuations of any other agent.

From this we show that if the set of public outcomes is finite, then a *bribe-proof* solution is, essentially, constant, in the sense that an agent's payoff is never affected by changes in any other agent's reported valuations. This result applies not only when all possible valuations of the public outcomes are admissible, but also when each agent's set of admissible valuations is a connected set. For the case of an infinite set of public outcomes, we derive the same conclusion as long as each agent's set of admissible valuations is "smoothly connected" in the sense of Holmström (1979), who generalizes the characterization of Clarke–Groves Mechanisms mentioned above to such domains.

Finally, we show that if the domain of valuation functions is sufficiently "rich", then a *bribe-proof* solution must actually be a constant function.

For the case of exactly two public outcomes (and the unrestricted domain of valuation functions), Green and Laffont (1979) consider manipulations by coalitions of a fixed size in which members make *joint* misrepresentations, along with transfers among themselves. They show that no Clarke–Groves Mechanism is immune to such manipulation by coalitions of any fixed size less than the total number of agents. Our result for finite sets of public outcomes is a substantial strengthening of their result for the particular case of manipulation by coalitions with a size of two.

2 Notation

There is a finite set of **agents**, $N = \{1, 2, \ldots, n\}$, $n \geq 2$, with arbitrary elements i and j. There is a compact set of **public outcomes**, Y, with arbitrary elements y and y'. In addition to the public outcome, each agent $i \in N$ consumes some amount $m_i \in \mathbb{R}$ of a **divisible good**, say money. An **allocation** consists of a public outcome $y \in Y$ and the specification of an amount of money for each agent, $m = (m_1, m_2, \ldots, m_n)$.

Each agent $i \in N$ has a quasi-linear (*i.e.* linearly additively separable) preference ordering over $Y \times \mathbb{R}$, parameterized by a set, $\Theta_i \subseteq \mathbb{R}^k$, of admissible

types.[2] That is, for each $i \in N$, there is a **valuation function**, $v_i: Y \times \Theta_i \to \mathbb{R}$, representing the preferences of an agent, depending on his type: an agent of type $\theta_i \in \Theta_i$ (weakly) prefers a bundle $(y, m_i) \in Y \times \mathbb{R}$ to another bundle (y', m_i') if and only if $v_i(y; \theta_i) + m_i \geq v_i(y'; \theta_i) + m_i'$.

Assumption 1 (Continuity of valuation functions) For all $i \in N$, v_i is continuous in $Y \times \Theta_i$.

We refer to $\Theta \equiv \Theta_1 \times \cdots \times \Theta_n$ as the **domain**. For any $i \in N$, the notation $\theta_{-i} \in \Theta_{-i}$ refers, as usual, to a list of types for agents other than i.

The results of Section 3.2 apply to domains that are connected, in the sense that if two types are admissible for an agent, then there is a path of admissible types connecting the two.

Connected: The domain Θ is connected if for all $i \in N$, Θ_i is a path-connected set: For all $\theta_i, \theta_i' \in \Theta_i$, there exists a continuous function $f: [0, 1] \to \Theta_i$ such that $f(0) = \theta_i$ and $f(1) = \theta_i'$.

A **solution** is a function $\varphi: \Theta \to Y \times \mathbb{R}^n$ choosing an allocation for any profile of admissible types. It will be notationally convenient to decompose the solution into two functions, $\bar{y}: \Theta \to Y$ and $\bar{m}: \Theta \to \mathbb{R}^n$, in which case we write $\varphi \equiv (\bar{y}, \bar{m})$. It will also be convenient to write for any agent $j \in N$, $\varphi_j(\theta) \equiv (\bar{y}(\theta), \bar{m}_j(\theta))$ to refer to agent j's consumption bundle.

Depending on the interpretation of the model, one may want to impose certain feasibility-type conditions on a solution, such as weak budget balance (for all $\theta \in \Theta$, $\sum \bar{m}_i(\theta) \leq M$, where M is some aggregate endowment of money), or strong budget balance ($\sum \bar{m}_i(\theta) = M$). Such requirements have no effect on our results, so we will not address them.

2.1 The Bribing Condition

If the agents are reporting their types to a planner who is using a given solution, it is of interest to know whether the solution satisfies certain incentive compatibility properties. For instance, it is desirable for a solution to be such that an agent of type θ_i can do no better for himself than by reporting the true type θ_i to the planner, regardless of the other agents' types.

[2] When Y is finite, you could imagine the set of types being $\mathbb{R}^{|Y|-1}$, and that a type represents an agent's normalized valuations of the public outcomes.

Strategy-proof: The solution $\varphi = (\bar{y}, \bar{m})$ is *strategy-proof* if for all $\theta \in \Theta$ and all $i \in N$, there exists no $\theta'_i \in \Theta_i$ such that

$$v_i(\bar{y}(\theta'_i, \theta_{-i}); \theta_i) + \bar{m}_i(\theta'_i, \theta_{-i}) \; > \; v_i(\bar{y}(\theta); \theta_i) + \bar{m}_i(\theta)$$

Holmström (1979) shows that on most connected domains, the only *strategy-proof* solutions that maximize $\sum_i v_i(y; \theta_i)$ for every profile θ are Clarke–Groves Mechanisms.

As discussed in the Introduction, we also wish to rule out the possibility that an agent could bribe another to misrepresent his type. We formulate a condition that rules out this type of situation.

Bribe-proof: The solution $\varphi = (\bar{y}, \bar{m})$ is *bribe-proof* if for all $\theta \in \Theta$ and all $i, j \in N$, there exists no $b \in \mathbb{R}$ and $\theta'_i \in \Theta_i$ such that

$$\begin{aligned} v_i(\bar{y}(\theta'_i, \theta_{-i}); \theta_i) + \bar{m}_i(\theta'_i, \theta_{-i}) + b \; &> \; v_i(\bar{y}(\theta); \theta_i) + \bar{m}_i(\theta) \\ v_j(\bar{y}(\theta'_i, \theta_{-i}); \theta_j) + \bar{m}_j(\theta'_i, \theta_{-i}) - b \; &> \; v_j(\bar{y}(\theta); \theta_j) + \bar{m}_j(\theta) \end{aligned}$$

Here, j bribes i with b units of money to misrepresent his type. Notice that by choosing $i = j$ and $b = 0$, *bribe-proofness* implies *strategy-proofness*.[3]

At this point, a few points are worth mentioning. First, we are implicitly assuming that the two agents would trust each other in arranging this misrepresentation; that is, j would not break his promise to pay i, and i would not reneg on his promise to misrepresent. This assumption follows in what Tirole (1992) calls the "enforceability approach".[4]

Second, note that we do not allow j to also misrepresent his type. Since we are determining the consequences of our condition (which in some instances are strong), the results are stronger using our weaker definition.

Third, we are implicitly assuming that the divisible good is perfectly transferable among agents. Perhaps instead, when j sends b units of money, i only receives $B(b) < b$ of it (*e.g.* due to some variable transaction cost). All of our results continue to hold as long as B is a monotonic function (satisfying

[3]One might argue that the *bribe-proofness* condition should be defined without implying *strategy-proofness* (*i.e.* disallowing $i = j$). However it would be unreasonable to attempt to rule out bribing situations while allowing the possibility that an agent could gain by simply misrepresenting his own type.

[4]See that paper for a justification of the assumption.

$B(0) = 0$, of course).

3 Results

Consider the following example of a domain for which there exist non-trivial, *bribe-proof* solutions.

Example 1 (Single-peaked preferences.) Let $Y = [a, b] \subset \mathbb{R}$ be an interval of public outcomes. For all $i \in N$, let $\Theta_i = \mathbb{R}$, and for all $y \in Y$ and all $\theta_i \in \Theta_i$, let $v_i(y, \theta_i) = -|y - \theta_i|$. Note that each $v_i(\cdot, \theta_i)$ is a single-peaked function on Y and that Θ is a connected domain.

A *bribe-proof* solution $\varphi = (\bar{y}, \bar{m})$ can be constructed by letting $\bar{m}$ be constant and letting $\bar{y}$ be defined as a median voter rule (Moulin, 1980), *e.g.*, if $|N|$ is odd, let $\bar{y}(\theta)$ be the median of $\theta_1, \ldots, \theta_n$.

We will point out particular attributes of this example in Sections 3.2–3.3. First, however, note that under any solution defined in Example 1, an agent's payoff varies continuously as any other agent varies is reported type. We first show that this is a general property of *bribe-proof* solutions.

3.1 Continuity

Since we are dealing with the behavior of *bribe-proof* solutions in response to changes in one agent's type, **for the remainder of the paper, we fix the following**,

- an agent $i \in N$ (potentially the bribee),
- types $\theta_{-i} \in \Theta_{-i}$ of the other agents,
- an agent $j \in N \setminus \{i\}$ (potentially the briber), and
- a *bribe-proof* solution $\varphi \equiv (\bar{y}, \bar{m})$.

The set of bundles that agent i can obtain by varying his type is his **option set**:

$$\bar{O}_i = \{(y, m_i) \in Y \times \mathbb{R} : \exists \theta_i \in \Theta_i \text{ such that } (y, m_i) = \varphi_i(\theta_i, \theta_{-i})\}$$

For all $\theta_i \in \Theta_i$, define the maximum payoff that i may receive, and his set of best obtainable bundles, as follows.

$$u_i^*(\theta_i) = \max_{(y,m_i)\in\bar{O}_i} v_i(y;\theta_i) + m_i \tag{1}$$

$$O_i^*(\theta_i) = \{(y,m_i) \in \bar{O}_i : v_i(y;\theta_i) + m_i = u_i^*(\theta_i)\} \tag{2}$$

Since φ is *strategy-proof*, u_i^* is well-defined; in fact, for all $\theta_i \in \Theta_i$ we have

$$u_i^*(\theta_i) = v_i(\bar{y}(\theta_i,\theta_{-i});\theta_i) + \bar{m}_i(\theta_i,\theta_{-i})$$

We first show that since φ is *strategy-proof*, u_i^* must be a continuous function.[5] If Y is finite, this follows from a direct application of the Maximum Theorem. However, in the general case, the option set, $\bar{O}_i$, may not be compact. Lemmas 1 and 2 show the boundedness of an agent's option set.

Lemma 1 *Suppose that Y is compact and φ is* strategy-proof. *There exists $M \subset \mathbb{R}$ such that (1) $\bar{O}_i \subseteq Y \times M$ and (2)* $\sup M < \infty$.

Proof: Suppose by contradiction that $\bar{O}_i$ contains a sequence of bundles $(y^k, m_i^k)_{k=1}^{\infty}$ such that $m_i^k \to \infty$. Let $\theta_i \in \Theta_i$ be such that $\varphi(\theta_i,\theta_{-i}) = (y^1, m_i^1)$.

By *strategy-proofness*, for all $k \in \mathbb{N}$, we have

$$v_i(y^1;\theta_i) + m_i^1 - (v_i(y^k;\theta_i) + m_i^k) \geq 0$$

Since $m_i^k \to \infty$, it must be that $v_i(y^k;\theta_i) \to -\infty$. However, since v_i is continuous and Y is compact, $v_i(\,\cdot\,;\theta_i)$ attains a (finite) minimum on Y, which is a contradiction. □

That lemma bounded agent i's option set from above. The additional (temporary) assumption of a compact type-space bounds it from below.

Lemma 2 *Suppose that Y is compact, Θ_i is compact, and φ is* strategy-proof. *There exists $M \subset \mathbb{R}$ such that (1) $\bar{O}_i \subseteq Y \times M$ and (2)* $\inf M > -\infty$.

[5] This general notion that *strategy-proofness* implies some sort of continuity has been accepted by some as "folk knowledge"; however it usually requires an assumption on the consumption space, such as our compactness assumption on Y.

Proof: Suppose by contradiction that $\bar{O}_i$ contains a sequence of bundles $\{(y^k, m_i^k)\}_{k=1}^{\infty}$ such that $m_i^k \to -\infty$. For all $k \in \mathbb{N}$, let $\theta^k \in \Theta_i$ be such that $\varphi_i(\theta^k, \theta_{-i}) = (y^k, m_i^k)$.

Since Θ_i is compact, the sequence $\{\theta^k\}$ contains a subsequence, $\{\hat{\theta}^\ell\}$, that converges to some $\theta_i \in \Theta_i$. Let $(y, m_i) = \varphi_i(\theta_i, \theta_{-i})$. By *strategy-proofness*, for all $\ell \in \mathbb{N}$, we have $v_i(y; \hat{\theta}^\ell) + m_i - (v_i(y^\ell; \hat{\theta}^\ell) + m_i^\ell) \leq 0$. Since $v_i(y; \hat{\theta}^\ell) \to v_i(y; \theta_i)$ and $m_i^\ell \to -\infty$, it must be that $v_i(y^\ell; \theta^\ell) \to \infty$. However since v_i is continuous and $Y \times \Theta_i$ is compact, v_i attains a (finite) maximum on $Y \times \Theta_i$, which is a contradiction. □

Therefore, agent i's option set on a compact domain is a bounded set. However, it may be open. To deal with that technical difficulty, define the following for all $\theta_i \in \Theta_i$.[6]

$$u^{**}(\theta_i) = \max_{(y, m_i) \in \text{cl}(\bar{O}_i)} v_i(y; \theta_i) + m_i \tag{3}$$

$$O^{**}(\theta_i) = \{(y, m_i) \in \text{cl}(\bar{O}_i) : v_i(y; \theta_i) + m_i = u^{**}(\theta_i)\} \tag{4}$$

Proposition 1 *Suppose that* Y *is compact,* Θ_i *is compact, and* φ *is* strategy-proof. *Then* u_i^* *is continuous in* Θ_i.

Proof: Lemmas 1 and 2 imply that $\text{cl}(\bar{O}_i)$ is compact. Therefore the Maximum Theorem implies that u^{**} is continuous and O^{**} is upper semi-continuous (u.s.c.).[7] Note that by the definition of *strategy-proofness*, u_i^* is in fact well-defined. Therefore, since v_i is continuous on $\text{cl}(\bar{O}_i)$, we have $u_i^* \equiv u^{**}$. Hence u_i^* is continuous. □

Note that since u_i^* is continuous on any compact (sub)domain, it is continuous on any domain (see the proof of Corollary 1).

For all $\theta_i \in \Theta_i$, define the payoff that j receives as follows.

$$u_j^*(\theta_i) = v_j(\bar{y}(\theta_i, \theta_{-i}); \theta_j) + \bar{m}_j(\theta_i, \theta_{-i}) \tag{5}$$

Now we have the main result.

Theorem 1 *Suppose that* Y *is compact,* Θ_i *is compact, and* φ *is* bribe-proof. *Then* u_j^* *is continuous in* Θ_i.

[6] The closure of a set S is denoted $\text{cl}(S)$.
[7] See Berge (1963) for a definition. The notation follows Sundaram (1996).

Proof: Suppose by contradiction that u_j^* is not continuous at some $\bar{\theta}_i \in \Theta_i$.

Case 1: There exists a sequence $\{\theta_i^k\}_{k=1}^{\infty}$ converging to $\bar{\theta}_i$, and $\bar{\epsilon} > 0$, such that for all k, $u_j^*(\bar{\theta}_i) - u_j^*(\theta_i^k) > \bar{\epsilon}$.

Let $(y, m_i) = \varphi_i(\bar{\theta}_i, \theta_{-i})$. *Bribe-proofness* implies that for all k, $u_i^*(\theta_i^k) - v_i(y; \theta^k) - m_i \geq \bar{\epsilon}$. However, Proposition 1 implies that $u_i^*(\theta_i^k)$ converges to $u_i^*(\bar{\theta}_i)$, and the continuity of v implies that $v_i(y; \theta^k)$ converges to $v_i(y; \bar{\theta}_i)$. Hence $u_i^*(\theta_i^k) - v_i(y; \theta^k) - m_i$ converges to 0, which is a contradiction.

Case 2: There exists a sequence $\{\theta_i^k\}_{k=1}^{\infty}$ converging to $\bar{\theta}_i$, and $\bar{\epsilon} > 0$, such that for all k, $u_j^*(\theta_i^k) - u_j^*(\bar{\theta}_i) > \bar{\epsilon}$.

Since, as above, O^{**} is u.s.c. and $u_i^* \equiv u^{**}$, we have for all $\theta_i \in \Theta_i$, $O_i^*(\theta_i) \subseteq O^{**}(\theta_i)$.

Since v_i is continuous, there exists an open set $\hat{O} \supset O^{**}(\bar{\theta}_i)$ such that $(y, m_i) \in \hat{O}$ implies $u_i^*(\bar{\theta}_i) - v_i(y; \bar{\theta}_i) - m_i < \bar{\epsilon}$. Since O_i^{**} is u.s.c., there exists k (sufficiently large) such that $O^{**}(\theta_i^k) \subset \hat{O}$. This implies $(\bar{y}(\theta_i^k), \bar{m}_i(\theta_i^k)) \in \hat{O}$. Therefore, $v_i(\bar{y}(\theta_i^k); \bar{\theta}_i) + \bar{m}_i(\theta_i^k) + \bar{\epsilon} > u_i^*(\bar{\theta}_i)$, contradicting *bribe-proofness*.
□

Finally, note that the assumption of a compact type-space was only needed temporarily.

Corollary 1 *Suppose that Y is compact and φ is* bribe-proof. *Then u_j^* is continuous in Θ_i.*

Proof: Suppose not. Then u_j^* violates continuity on some compact subdomain, $\bar{\Theta}_i \subset \Theta_i$. The restriction of φ to the subdomain $\bar{\Theta}_i \times \Theta_{-i}$ defines a discontinuous *bribe-proof* solution (on a new domain) violating the conditions of Theorem 1. □

We end this section by noting that the assumption of a compact Y can not simply be dropped.

Example 2 Let $Y = [0, \infty)$, $\Theta_1 = [0, 1]$, and v_1 satisfy

$$v_1(y, \theta_1) = \begin{cases} -y & \text{if } \theta_1 = 0, \\ \max\{-y, 2 - |\frac{1}{\theta_1} - y|\} & \text{if } \theta_1 > 0. \end{cases}$$

Let $\Theta_2 = \{\theta_2\}$, and $v_2(y;\theta_2) \equiv 0$. Let $\hat{\varphi}$ satisfy

$$\hat{\varphi}(\theta_1, \theta_2) = (\bar{y}(\theta_1), \bar{m}_1(\theta_1), \bar{m}_2(\theta_1)) = \begin{cases} (0,0,0) & \text{if } \theta_1 = 0, \\ (\frac{1}{\theta_1}, -\frac{1}{2}, \frac{1}{2}) & \text{if } \theta_1 > 0. \end{cases}$$

One may check that $\hat{\varphi}$ is *bribe-proof*, but that neither u_1^* nor u_2^* is continuous at $\theta_1 = 0$. A similar example can be constructed in which Y is bounded but open.

3.2 Finite Sets of Public Outcomes

Consider again the *bribe-proof* solutions in Example 1. Suppose that instead of being an interval, Y is a finite subset of $\mathbb{R}$, *e.g.* $Y = \{1, 2, 3\}$. Median-voter types of solutions are well-defined in this setting also, subject to some tie-breaking procedure when the "median voter" is indifferent between two elements of Y. However such solutions are no longer *bribe-proof*! Informally, when the median voter is indifferent (or almost indifferent) between two elements of Y, he could be bribed to misreport his preferences.

A trivial example of a *bribe-proof* solution for this environment is a constant solution. The discouraging news is that for connected domains, if Y is finite, then this is essentially the *only* type of *bribe-proof* solution. The following example illustrates why it is only *essentially* so.

Example 3 Let $N = \{1, 2\}$, $Y = \{a, b\}$, $\Theta_1 = \mathbb{R}$, $\Theta_2 = \{1\}$, $v_1(a, \theta_1) = v_1(b, \theta_1) + \theta_1$ for all θ_1, and $v_2(a, 1) = v_2(b, 1) + 1$. Note that Θ is connected, and that the following solution is *bribe-proof*. For all $\theta \in \Theta$, let

$$\hat{\varphi}(\theta) = \begin{cases} (a, 0, 0) & \text{if } v_1(a, \theta_1) \geq v_1(b, \theta_1) - 1 \\ (b, -1, 1) & \text{otherwise} \end{cases}$$

For the solution in Example 3, agent 1 plays the role of a dictator, but agent 2, who only has one type, does not care what agent 1 picks — agent 2 is always indifferent between the two bundles he receives. For connected domains and finite Y, this is the only way a *bribe-proof* solution may not be constant — when the solution is responsive to a change in an agent's type (agent 1 in Example 3), the change is trivial to all other agents.

Theorem 2 *Suppose that Y is finite, Θ is a connected domain, and φ is* bribe-proof. *Then u_j^* is constant in Θ_i.*

Proof: Since Y is finite, so is $\bar{O}_i$.[8] Therefore, O_i^* is finite, and the Maximum Theorem directly implies O_i^* is u.s.c. (without the assumption that Θ_i is compact, as needed above). Hence for all $\theta_i \in \Theta_i$ there exists $\delta > 0$ such that $|\theta_i' - \theta_i| < \delta$ implies $O_i^*(\theta_i') \subseteq O_i^*(\theta_i)$. Therefore for any such θ_i', we have $v_i(\bar{y}(\theta_i', \theta_{-i}); \theta_i) + \bar{m}_i(\theta_i', \theta_{-i}) = u_i^*(\theta_i)$. Therefore *bribe-proofness* implies $u_j^*(\theta_i') \leq u_j^*(\theta_i)$ (otherwise j would bribe i).

We have shown that each $\theta_i \in \Theta_i$ is a local maximizer of u_j^*. Therefore, since Θ_i is path-connected and u_j^* is continuous (Corollary 1, u_j^* is constant (see Lemma 3 in the Appendix). □

Theorem 2 actually implies that under a *bribe-proof* solution, each agent is actually a dictator on the range of the solution. Formally, the solution must satisfy the following condition.

***All-Dictatorial*:** The solution $\hat{\varphi} = (\bar{y}, \bar{m})$ is *all-dictatorial* if for all $\theta, \theta' \in \Theta$ and all $i \in N$, we have $v_i(\bar{y}(\theta); \theta_i) + \bar{m}_i(\theta) \geq v_i(\bar{y}(\theta'); \theta_i) + \bar{m}_i(\theta')$.

The Corollary follows directly from Theorem 2.

Corollary 2 *Suppose that Y is finite and Θ is a connected domain. Then φ is* bribe-proof *if and only if φ is* all-dictatorial.

A relevant concept here is Hurwicz and Walker's (1990) notion of a "decomposable" domain. When each agent cares only about his own dimension of the outcome space (*i.e.*, the domain is decomposable), it is a trivial matter to define *bribe-proof* solutions. They can even be efficient, but they need not be anything close to "constant". However when there is a "conflict of interest" between agents (and the domain is indecomposable), the condition (*all-dictatorial*) is much stronger. For example, in Section 3.4 we examine a class of domains for which this conflict of interests always exists, and derive an even stronger conclusion.

The next result follows from Corollary 2.

[8] Recall that by *strategy-proofness*, $(y, m_i), (y, m_i') \in \bar{O}_i$ implies $m_i = m_i'$.

Corollary 3 *Suppose that Y is finite, Θ is a connected domain, and φ is* bribe-proof. *For all $\theta, \theta' \in \Theta$, if $\varphi(\theta) = (y, m)$ and $\varphi(\theta') = (y, m')$, then $m = m'$.*

Before concluding this section, note that even if a domain of interest is not connected, the results could be applied to each "connected component" of the domain, that is, each subdomain that itself forms a connected domain.

3.3 Smooth Preferences

Consider once again the solutions described in Example 1. Note that each single-peaked valuation function has a slope of 1 to the left of its peak and a slope of -1 to the right. As we did originally, let $Y = \mathbb{R}$, but now suppose that the domain was such that "steeper" and "flatter" single-peaked valuation functions were also admissible. Again, the median-voter types of solutions are still well-defined in such a setting. However, they are not *bribe-proof.* If the median voter has a relatively flat valuation function, he cares less about the location of y, relative to his transfer, than agents with steeper valuation functions, and can be bribed. Similarly, if valuation functions were smooth, the same problem would arise: The median voter would have a "locally flat" valuation function at his peak, and could be bribed to make at least a small misrepresentation.

In fact, Theorem 2 and Corollary 2 generalize to the case in which Y is infinite, as long as the domain is "smoothly" connected, in the sense of Holmström (1979). That is, there should exist a smooth, one-dimensional parameterization of some path between any two types:

Smoothly Connected: The domain Θ is smoothly connected if for all $i \in N$ and all $\theta_i, \theta_i' \in \Theta_i$, there exists $w: Y \times [0, 1] \to \mathbb{R}$ such that

i. For all $x \in [0, 1]$, there exists $\theta_i^x \in \Theta_i$ such that $w(\,\cdot\,, x) = v_i(\,\cdot\,; \theta_i^x)$

ii. $w(\,\cdot\,, 0) = v_i(\,\cdot\,; \theta_i)$

iii. $w(\,\cdot\,, 1) = v_i(\,\cdot\,; \theta_i')$

iv. For all $y \in Y$, $w(y, \cdot)$ is differentiable on $[0, 1]$

v. There exists $z \in \mathbb{R}$ such that for all $y \in Y$ and all $x \in [0, 1]$,

$$\left| \frac{\partial w(y, x)}{\partial x} \right| \leq z$$

One may check that the domain of Example 1 can not be parameterized in this way, so it is not smoothly connected. On the other hand, if v_i is differentiable in Θ_i for all $i \in N$, then any convex θ is smoothly connected.

Theorem 3 *Suppose that* Θ *is a smoothly connected domain and* φ *is* bribe-proof. *Then* u_j^* *is constant in* Θ_i.

Proof: As in the definition of smoothly connected domains, let w be defined with respect to Θ_i, and for all $x \in [0, 1]$, let θ_i^x be defined as in (i).

Define $f: [0, 1]^2 \to \mathbb{R}$ so that for all $x, x' \in [0, 1]$,

$$f(x, x') \equiv v_i(\bar{y}(\theta_i^x, \theta_{-i}), \theta_i^{x'}) + \bar{m}_i(\theta_i^x, \theta_{-i})$$

Strategy-proofness implies that for all $x' \in [0, 1]$,

$$x' \in \arg\max_{x \in [0,1]} f(x, x') \tag{6}$$

Bribe-proofness implies that for all $x' \in [0, 1]$,

$$x' \in \arg\max_{x \in [0,1]} f(x, x') + u_j^*(\theta_i^x) \tag{7}$$

Since Θ is smoothly connected, the Lemma in the Appendix of Holmström (1979) states that (6) and (7) imply that u_j^* is constant. □

As in the previous section, this result can be used to derive the following.

Corollary 4 *Suppose that* Θ *is a smoothly connected domain. Then* φ *is* bribe-proof *if and only if* φ *is* all-dictatorial.

3.4 Rich Domains

Finally, one might observe that in the above examples, in which non-constant *bribe-proof* solutions exist, the domains are, loosely speaking, "narrow". For example, they don't contain many perturbations of the functions they contain. It turns out that if a domain is rich enough, then in fact only constant functions are *bribe-proof*.

We will use the following definition of richness, which requires that if a valuation function is admissible, then for any outcome $y \in Y$, there exists another admissible valuation function for which the value of y, relative to any other outcome, is strictly greater than for the original function.

Monotonically Closed: The domain Θ is monotonically closed if for all $i \in N$, all $\theta_i \in \Theta_i$, and all $y \in Y$, there exists $\theta'_i \in \Theta_i$ such that for all $y' \in Y \setminus \{y\}$, $v_i(y;\theta'_i) - v_i(y';\theta'_i) > v_i(y;\theta_i) - v_i(y';\theta_i)$.

Theorem 4 *Suppose that Y is finite, and that Θ is connected and monotonically closed. If φ is* bribe-proof, *then it is a constant function.*

Proof: Suppose by contradiction that there exist distinct (y, m) and (y', m') in the range of φ. By Corollary 3, $y \neq y'$, so without loss of generality we have $(y, m_i), (y', m'_i) \in \bar{O}_i$ with $y \neq y'$. In this proof we will change agent j's type (from θ_j). To simplify notation, let $\varphi = (\bar{y}, \bar{m})$ depend only on the types of agents i and j.

Let $\theta_i \in \Theta_i$ satisfy $\varphi(\theta_i, \theta_j) = (y, m)$. Since agent i receives (y', m'_i) for *some* reported type, and since Θ is monotonically closed, there exists $\theta'_i \in \Theta_i$ such that

$$\{(y', m'_i)\} = \arg\max_{(\hat{y}, \hat{m}_i) \in \bar{O}_i} v_i(\hat{y}; \theta_i) + \hat{m}_i \tag{8}$$

Strategy-proofness implies $\varphi_i(\theta'_i, \theta_j) = (y', m'_i)$. Corollary 3 therefore implies

$$\varphi(\theta'_i, \theta_j) = (y', m') \tag{9}$$

Corollary 2 implies that for all $\hat{\theta}_i, \hat{\theta}_j$, $v_j(y; \theta_j) + m_j \geq v_j(\bar{y}(\hat{\theta}_i, \hat{\theta}_j); \theta_j) + \bar{m}_j(\hat{\theta}_i, \hat{\theta}_j)$. So, since Θ is monotonically closed, there exists $\theta'_j \in \Theta_j$ such that for all $\hat{\theta}_i, \hat{\theta}_j$,

$$\varphi_j(\hat{\theta}_i, \hat{\theta}_j) \neq (y, m_j) \implies v_j(y; \theta'_j) + m_j > v_j(\bar{y}(\hat{\theta}_i, \hat{\theta}_j); \theta_j) + \bar{m}_j(\hat{\theta}_i, \hat{\theta}_j) \tag{10}$$

Strategy-proofness then implies $\varphi_j(\theta_i, \theta'_j) = (y, m_j)$, so by Corollary 3,

$$\varphi(\theta_i, \theta'_j) = (y, m) \tag{11}$$

Theorem 2 implies

$$v_j(\bar{y}(\theta'_i, \theta'_j); \theta'_j) + \bar{m}_j(\theta'_i, \theta'_j) = v_j(\bar{y}(\theta_i, \theta'_j); \theta'_j) + \bar{m}_j(\theta_i, \theta'_j)$$

With eqns. (10) and (11), this implies $\varphi_j(\theta_i', \theta_j') = (y, m_j)$, so by Corollary 3,

$$\varphi(\theta_i', \theta_j') = (y, m) \tag{12}$$

Similarly, Theorem 2 implies

$$v_i(\bar{y}(\theta_i', \theta_j'); \theta_i') + \bar{m}_i(\theta_i', \theta_j') = v_i(\bar{y}(\theta_i', \theta_j); \theta_i') + \bar{m}_i(\theta_i', \theta_j)$$

With eqns. (9) and (12), this implies $v_i(y; \theta_i') + m_i = v_i(y'; \theta_i') + m_i'$, contradicting eqn. (8). □

4 Conclusion

We have presented a model in which agents have quasi-linear preferences over outcomes and transfers, and shown that in many situations it is essentially impossible to design a solution immune to manipulation by pairs of agents. Throughout the discussion, the interpretation of the model was one in which an outcome represents a public decision, such as a level of public goods. However, there are also applications of this model to various private-goods environments. Examples include auctions, more general allocation problems with indivisible goods,[9] and various two-sided matching problems with money such as generalized assignment problems and many-to-one matching problems. In such environments, the "public" outcomes are actually allocations of the private goods to the agents.

Of additional interest is a seemingly related result by Crémer (1996) for the case of exactly two outcomes ($|Y| = 2$), which also concerns manipulation by pairs of agents, and is much less negative.

The remainder of the section is dedicated to an informal discussion of Crémer's result, and the way our two sets of results together establish a boundary between possibility and impossibility at the point where agents gain information about each others' types.

Crémer's setup is as follows. Imagine that a Clarke–Groves Mechanism is being used, and that all of the agents except, say, i and j have already revealed their valuation functions to the mechanism (so interpret them as fixed). Further, imagine that agents i and j do not know each other's valua-

[9] See Schummer (1997) for an application of Theorem 2 to such an environment.

tion functions, but anticipate the possibility of gains by both *jointly* misrepresenting their valuations *and* making an internal transfer, for certain realizations of their true valuation functions. Since they do not know each others' valuations, they coordinate their potential misrepresentation by devising a "sub-mechanism", to which they report their valuations, that determines for them (1) a (mis-)report of their valuation functions to be made to the original Clarke–Groves Mechanism, and (2) a transfer to be made between the two.

The question is whether, for a given Clarke–Groves Mechanism, a pair of agents could devise such a sub-mechanism that is *strategy-proof* (when interpreted as a 2-agent solution). The answer is *sometimes*: Crémer (1996) provides some Clarke–Groves Mechanisms that are immune to such manipulation by pairs of agents.[10]

Now one may think that Theorem 2 contradicts Crémer's result with the following reasoning: If a Clarke–Groves Mechanism is not *bribe-proof*, as shown by Theorem 2, why can we not design a sub-mechanism for some pair of agents, as above, to implement this bribe, violating the result of Crémer? The reply to this is that such a sub-mechanism would be manipulable by one of the two agents — one of the two agents cheating the system will be cheated by the other agent.

For a precise example, suppose φ is a *strategy-proof* solution (for example a Clarke–Groves Mechanism) that is not *bribe-proof*: given types for the agents $\theta \in \Theta$, suppose agent j can successfully bribe agent i with b units of money to mis-report his type as θ_i'. One may propose the following sub-mechanism for the two agents: If the other agents have reported those types θ_{-ij}, and if i and j report (θ_i, θ_j) to the sub-mechanism, then the sub-mechanism recommends the mis-report (θ_i', θ_j), and a transfer of b to be made from j to i. In all other cases, the sub-mechanism recommends no misrepresentation, and no transfer. Is this sub-mechanism *strategy-proof*?

If j is not of type θ_j, then i can not manipulate the sub-mechanism. Similarly, j can not if i is not of type θ_i, and these are almost all of the cases. However, consider the situation when their types are (θ_i', θ_j). Here, agent i can tell the sub-mechanism he is type θ_i; the sub-mechanism recommends the

[10] However, he shows that all Clarke–Groves Mechanisms are manipulable to this sort of manipulation by *triples* of agents. On the other hand, he shows that each such manipulation by a triple is, itself, re-manipulable by two of those agents!

report (θ_i', θ_j) to the Clarke Groves Mechanism, which is what would have been reported anyway, along with a transfer of b to agent i, resulting in a gain to agent i. Hence this is not a *strategy-proof* sub-mechanism.

Furthermore, Crémer's result tells us that for a particular class of Clarke–Groves Mechanisms, there is *no* way for a pair of agents to design a *strategy-proof* sub-mechanism taking advantage of any such bribing situation.

The essential difference between *bribe-proofness* and Crémer's manipulation condition is in the need for potential manipulators to know each other's types. Under the stronger condition of *bribe-proofness*, a manipulation is considered possible if there is any situation in which a pair of agents could gain through the bribing procedure. However, for a pair of agents to be able to gain with a sub-mechanism, they must devise a plan of manipulation that covers all realizations of their valuation functions, and it must be immune to further manipulation by any of the two individuals.

Since the two concepts are similar except for their respective implicit assumptions regarding the information agents have about each others' preferences, the two sets of results could be seen as a dividing line between the possibility and the impossibility of having solutions that are non-manipulable by coalitions of agents. Possibility obtains even among the class of Clarke–Groves Mechanisms as soon as potentially misrepresenting agents lose the information of each other's types. With perfect information, however, manipulation is possible under almost any solution, in many environments.

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Appendix

Lemma 3 *Let X be a path-connected set. If $f: X \to \mathbb{R}$ is continuous and if for all $x \in X$, x is a local maximizer of f, then f is constant.*

Proof: Suppose f is continuous and not constant. Then there exist $x, y \in X$ such that $f(x) < f(y)$. Let $g: [0,1] \to X$ be continuous, and satisfy $g(0) = x$ and $g(1) = y$. Let $L = \{\delta \in [0,1] : 0 \leq \delta' \leq \delta \implies f(g(\delta')) \leq f(x)\}$. (Note that L is a non-empty, connected set.) Let $\bar{\delta} \equiv \sup L$.

Since f is continuous, $\bar{\delta} \in L$, so $g(\bar{\delta})$ is not a local maximizer of f. □

The continuity of f can be replaced with every x also being a local minimizer: If f is not continuous, then $g(\bar{\delta})$ in the proof of the Lemma is either not a local maximizer or not a local minimizer. That is, if f is not constant, then either there exists a non-local-maximizer or there exists a non-local-minimizer, but not necessarily both. For example, consider the function $f(x) = 0$ for $x \neq 1$ and $f(1) = 1$: every x is a local maximizer.

Proof of Corollary 2

Corollary 2 *Suppose that Y is finite and Θ is a connected domain. Then φ is* bribe-proof *if and only if φ is* all-dictatorial.

Proof: Suppose by contradiction that for some $\theta, \theta' \in \Theta$ and $k \in N$,

$$v_k(\bar{y}(\theta); \theta_k) + \bar{m}_k(\theta) < v_k(\bar{y}(\theta'); \theta_k) + \bar{m}_k(\theta')$$

By repeated application of Theorem 2,

$$v_k(\bar{y}(\theta_k, \theta'_{-k}); \theta_k) + \bar{m}_k(\theta, \theta'_{-k}) < v_k(\bar{y}(\theta'); \theta_k) + \bar{m}_k(\theta')$$

contradicting *strategy-proofness*. □

Coalitional Strategy-proofness

It is simple to observe that any *all-dictatorial* solution is also *coalitionally strategy-proof*. Therefore on any domain of the types discussed in Sections 3.2–3.4, *bribe-proof* implies *coalitional strategy-proofness*. The following trivial example shows, however, that this logical relation does not hold in general.

Let $N = \{1, 2\}$, $Y = \{a, b, c, d\}$, and $\Theta_1 = \Theta_2 = \{0, 1\}$. Let the valuation functions satisfy:

	a	b	c	d
$v_1(\cdot; 0)$	0	2	−10	1
$v_1(\cdot; 1)$	1	−10	2	0
$v_2(\cdot; 0)$	0	−10	2	1
$v_2(\cdot; 1)$	1	2	−10	0

One *bribe-proof* solution that is not *coalitionally strategy-proof* is $\varphi = (\bar{y}, \bar{m})$, where $\bar{m}_1(\cdot) \equiv \bar{m}_2(\cdot) \equiv 0$, and

$$\bar{y}(0, 0) = a$$
$$\bar{y}(0, 1) = b$$
$$\bar{y}(1, 0) = c$$
$$\bar{y}(1, 1) = d$$

Also note that this rule is not efficient; hence *bribe-proofness* does not even imply *efficiency*.

Discussion Paper No. 1208

Capacity Investment under Demand Uncertainty: The Option Value of Subcontracting

by

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Abstract

We study the financial value of subcontracting by analyzing a competitive stochastic investment game with recourse. The manufacturer and subcontractor decide separately on their capacity investment levels. Then demand uncertainty is resolved and both parties have the option to subcontract when deciding on their production and sales. First, we study *price-only contracts* where an ex-ante transfer price is set for each unit supplied by the subcontractor. We characterize the sub-game perfect investment strategy and present an outsourcing condition. Manufacturer and supplier capacity levels are imperfect substitutes that, surprisingly, are more sensitive to changes in the cost structure than in the revenue structure. Uncertainty is the key reason, and we show that manufacturers will subcontract more when the level of market risk increases and when markets are more negatively correlated. As with financial options, this is accompanied by an increase in the option value of subcontracting. Second, we consider an *incomplete contract*, so that both parties negotiate over the subcontracting transfer. Depending on the manufacturer's "bargaining power," system performance can exceed that with price-only contracts. Finally, the third contract type is a *state-dependent price-only contract* for which we show an equivalence result with the bargaining contract. While subcontracting with these three contract types improves system performance, it cannot eliminate all decentralization costs (or "coordinate" the supply system).

Key Words: Real investments, capacity planning, subcontracting, outsourcing, supply contracts.

1 Introduction

We present analytic models to study subcontracting and outsourcing, two prevalent business practices across many industries. While the word *subcontracting* has been used for nearly two centuries, *outsourcing* first appeared in the English language only as recently as 1982 [2]. Both terms refer to the practice of one company (the subcontractor or supplier) providing a service or good for another (the contractor, buyer or manufacturer). Subcontracting typically refers to the situation where the contractor "procures an item or service which is normally capable of economic production in the contractor's own facilities and which requires the contractor to make specifications available to the subcontractor [7]." Outsourcing refers to the special case where the contractor has no in-house production capability and is dependent on the subcontractor for the entire product volume.

Many literatures discuss the costs and benefits of subcontracting. According to the strategy literature, subcontracting and outsourcing occur because a firm may find it unprofitable or infeasible to have all required capabilities in house: "a firm should concentrate on its core competencies and strategically outsource other activities [19]" and "not one company builds an entire flight vehicle, not even the simplest light plane, because of the exceptional range of skills and facilities required

[1]". Subcontracting and outsourcing may also be "an impetus and agent for change" and "may improve unduly militant or change-resisting" employee relations [4]. These benefits come at a cost by exposing the contractor to strategic risks, such as dependence on the subcontractor (with its inherent loss of control and associated hold-up risk) and vulnerability (e.g., lower barriers to entry and loss of competitive edge and confidentiality) [19]. The operations literature highlights the flexibility that subcontracting offers to production and capacity planning. Like demand and inventory management, subcontracting allows for short term capacity adjustments in the face of temporal demand variations. The key distinction between subcontracting and these other two production planning strategies however, is that subcontracting "requires agreement with a third party who may be a competing firm with conflicting interests [14]." (The implication being that any reasonable model of subcontracting must incorporate multiple decision makers.) From a financial perspective, the main reported benefits of subcontracting and outsourcing are lower operating costs and lower investment requirements for the contractor, and the spreading of risk between the two parties. Empirical studies report that cost efficiency is the prime motivation for outsourcing maintenance [4] and information systems [16]. It is also argued that contractors 'push the high risk' onto subcontractors by having them "carry a disproportionate share of market uncertainties [8]." The financial costs of subcontracting and outsourcing include decreased scale economies to the contractor [10] and the transaction costs resulting from the initiation and management of the contracting relationship [19]. Finally, an extensive economics literature discusses our topic when studying vertical integration but that literature generally ignores capacity considerations.

Few papers explicitly study an analytic model of subcontracting. Kamien and Li [14] present a multi-period, game theoretic aggregate planning model with given capacity constraints and show that the option of subcontracting results in production smoothing. Kamien, Li and Samet [15] study Bertrand price competition with subcontracting in a deterministic game with capacity constraints implicit in their convex cost structure. Hanson [11] develops and empirically tests a model of the optimal sharing of the ownership of a given, exogenously determined number of units of an asset between a manufacturer and a subcontractor. Tournas [20] captures asymmetries in in-house information in a principal-agent model and compares them with the bargaining cost of a captive outside contractor in a low-or-high demand scenario. Recently, Brown and Lee [5] have proposed a flexible reservation agreement in which a manufacturer may reserve supplier capacity in the form of options. Finally, there is significant literature on outsourcing in supply-chains. Cachon and Lariviere [6] give an excellent overview of various possible contract types and their costs and benefits, which will be discussed in more detail in Section 4.

The model presented in Section 2 below uses a two-stage, two-player, two-market stochastic game to examine the financial impact of the subcontracting option on capacity investment levels. In stage one, the manufacturer and subcontractor decide separately on their investment levels. Then demand uncertainty is resolved and both parties have the option to subcontract when deciding on their production levels in stage two, constrained by their earlier investment decisions. Subcontracting is viewed as a trade of the supplier's product for the manufacturer's money. We first analyze two scenarios (the centralized firm vs. two independent firms without any subcontracting) that give us performance references. In Section 3 we study *price-only contracts* where an ex-ante transfer price is set for each unit supplied by the subcontractor. We characterize the subgame perfect investment strategy and formulate an outsourcing threshold condition in terms of the manufacturer's investment cost. We show that optimal manufacturer and supplier capacity levels are imperfect substitutes with respect to capacity costs and contribution margins. Surprisingly, optimal capacity levels are more sensitive to changes in the cost structure (i.e., capacity costs) than in the revenue structure (i.e., margins or output prices). Uncertainty is the key reason. We also show that manufacturers will indeed subcontract more when the level of market uncertainty (risk) increases and when markets are more negatively correlated. This is accompanied by an increase in the option value of subcontracting (real assets), similar to the option value of financial assets. In Section 4 we study two other contract types. One uses the *incomplete contracting approach* where no explicit contracts can be made and both parties negotiate over the subcontracting transfer. This is the ultimate minimalistic and opportunistic approach to subcontracting. It allows us to analyze the role of the "bargaining power" of the contractor on outsourcing decisions and system performance improvement (which may be greater than with price-only contracts). Our third contract type is a *state-dependent price-only contract* for which we show an equivalence result with the bargaining contract. While subcontracting with any of these three contract types improves system performance relative to the independent scenario, it cannot eliminate all decentralization costs (or "coordinate" the supply system) due to uncertainty. We close with a discussion of more complex contracts in the literature and suggestions for further work.

Our model differs from those in the previous papers in that the capacity investment levels of both the manufacturer and the subcontractor are decision variables. Our multi-variate, multi-dimensional competitive newsvendor formulation is an extension of univariate, one-dimensional supply models and of the univariate competitive newsvendor models of Li [17] and Lippman and McCardle [18]. Our multi-dimensional model allows us to study the impact of subcontracting on both players' in-house investment levels and on the buyer's outsourcing decision, which is pre-assumed in captive-buyer captive-supplier models. We show that the higher complexity of subcontracting makes coordination more difficult compared to traditional outsourcing models in supply chains. The multi-variate demand distribution allows us to investigate the important role of market demand correlation and provides a graphical interpretation of the solution. Finally, we have chosen to make both models essentially single-period and to posit no information asymmetries between the two parties. Therefore we shall not discuss how subcontracting can smooth production plans over time, create or mitigate information asymmetry problems, or affect the long-run competitive position of the firms.

2 A Subcontracting Model

2.1 The Model

Consider a two-stage stochastic linear model of the investment decision process of two firms. In stage one when market demands are still uncertain, both firms must decide separately, yet simultaneously, on their capacity investment levels. At the beginning of stage two, market demands are realized and both firms must decide on their production levels to satisfy optimally market demands, constrained by their earlier investment decisions. At this stage, both firms have the option to engage in a trade. The subcontractor S can supply the manufacturer M a quantity $x_t \geq 0$ in exchange for a payment $p_t x_t$. Before we explain the specifics of the supply contract in the next section, let us discuss model features, notation and two reference scenarios that are useful to evaluate the impact of subcontracting on firm performance.

In the first reference scenario subcontracting is not an option ($x_t = 0$) so that both firms operate completely *independent* of each other. Both firms *go solo* and each will sell to its own market as shown in Figure 1. For simplicity, we will assume that both firms have exclusive access to their respective markets. Because the subcontractor lacks the assembly, marketing and sales clout of the manufacturer, she does not have direct access to market M. In practice, however, the manufacturer may have access to market S through wholly owned upstream subsidiaries that provide them and others with parts or subsystems. General Motors, for example, owns Delphi Automotives which supplies GM and other auto assemblers with brake systems and other parts. At the same time, GM

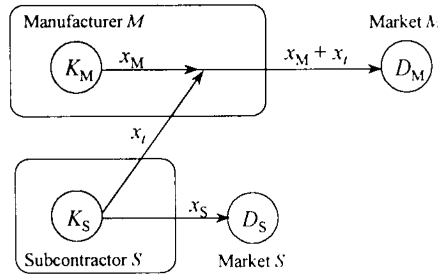

Figure 1: The Subcontracting Model.

multisources some parts from outside, independent subcontractors. Thus, market M would represent the end market for cars and market S the intermediate market for parts. GM could compete in market S but we will abstract from such competition to highlight the subcontracting option. Also, notice that direct sourcing from market S instead of from the subcontractor is not an option for the manufacturer. This modeling assumption reflects the relationship-specific information typically present in subcontracting and it implies that we are not discussing the purchase of standardized, off-the-shelve products in commodity markets.

The second reference scenario represents the other extreme in which both firms are *integrated* and controlled by a single decision maker. In this *centralized* scenario the integrated firm will serve both markets. Subcontracting, then, is the intermediate scenario in which both firms are independently owned so that we have two decision makers, yet trading is possible. (Thus the subcontractor's technology is sufficiently flexible that it can produce the same product as the manufacturer's technology.)

Let $K_i \geq 0$ denote firm i's capacity investment level, where $i = M$ or S. Firm i is assumed to face a constant marginal investment cost $c_i > 0$, so that its capacity investment cost $c_i K_i$ is linear in the investment level. The manufacturer's production level x_M and the supplier's production $x_S + x_t$ are linearly constrained by the capacity investment levels: $x_M \leq K_M$ and $x_S + x_t \leq K_S$. For simplicity, we assume that both firms make constant unit contribution margins p_i per unit sold in market i. To avoid trivial solutions we assume that $c_i < p_i$. Let $D_i \geq 0$ denote the product demand in market i. Like Kamien and Li [14], we assume symmetric information in the sense that each firm has complete information about the other's cost and profit structure and investment level, and they share identical beliefs regarding future market demands. (They have the same available market information or use the same forecasting method.) Beliefs regarding future market demands can then be represented by a single, multi-variate probability measure $P(\cdot)$. For simplicity, we assume that market demands are finite with probability one and that P has a continuous density $f(\cdot)$ on the sample space $\mathbb{R}^2_+$. The expectation operator will be denoted by E. We assume zero shortage costs and zero salvage values for both products and production assets[1]. Finally, both firms are assumed to be expected profit maximizers and the research question can thus be formulated in the two reference scenarios as follows.

2.2 Independents: Going Solo

When both firms do not subcontract, each firm decides on its production and sales decision x_i in stage two by maximizing its revenue $\pi_i = p_i x_i$ subject to the capacity constraint $x_i \leq K_i$ and the

[1]Relaxation of these assumptions to include convex investment costs, market-and-firm specific unit contribution margins p_{ij}, shortage costs and salvage values, and non-unit capacity consumption rates is relatively straightforward at the expense of added notational complexity.

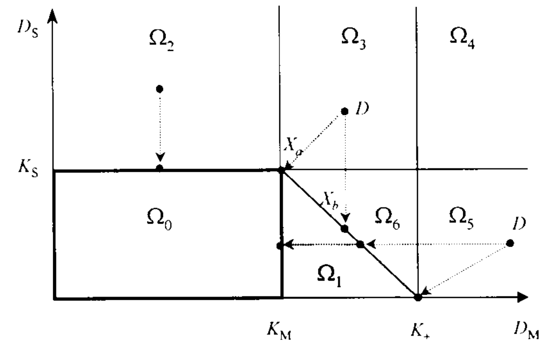

Figure 2: Production decisions and total market supply vector X, represented by arrows, depend on the demand D realization and the scenario.

demand constraint $x_i \leq D_i$. This "product mix" linear program can be solved by inspection to yield the optimal activity level and revenue: $x_i^{solo} = \min(K_i, D_i)$ and $\pi_i^{solo} = p_i x_i^{solo}$. In stage 1, firm i chooses its optimal investment level K_i^{solo} so as to maximize its expected firm value, denoted by V_i, which is the expected operating profit minus investment costs:

$$K_i^{solo} = \arg\max_{K_i \geq 0} V_i^{solo}(K) \qquad \text{where} \qquad V_i^{solo}(K) = \mathrm{E}\pi_i^{solo}(K, D) - c_i K_i. \tag{1}$$

The optimal solution is given by a simple newsvendor solution $K_i^{solo} = G_i\left(\frac{c_i}{p_i}\right)$, where $G_i(\cdot)$ is the inverse of the marginal tail probability function: $P(D_i > G_i(x)) = x$. To build some intuition for the solution technique that will be used below, let us summarize briefly how this familiar result can be derived using the multi-dimensional newsvendor model of Harrison and Van Mieghem [12]. It will be convenient to partition the demand space as in Figure 2 (where we abbreviated the sum of the components of K by $K_+ = K_1 + K_2$). Any given capacity vector K partitions $\mathbb{R}_+^2$ into 7 regions $\Omega_l(K), l = 0, 1, \cdots, 6$. The rectangular region $\Omega_0(K)$ is the capacity region of this two-firm supply system without subcontracting. Whenever D is within the supply system's capacity region, all market demand can be met. Outside the capacity region, some demand will be lost and the optimal production-subcontracting market supply $X = (x_M, x_S) \leq D$, represented by an arrow emanating from D, will be on the capacity frontier.

Linear programming theory yields that the revenue vector $\pi^{solo}(K, D)$ is unique and concave in K. Thus, the linear superposition $\mathrm{E}\pi_i^{solo}(K, D)$ and thus $V_i^{solo}(\cdot)$ are also concave so that the first order conditions of (1) are sufficient:

$$\frac{\partial}{\partial K_i} V_i^{solo} = -\nu_i^{solo} \text{ and } \nu_i^{solo} K_i^{solo} = 0,$$

where $\nu_i^{solo} \geq 0$ is the optimal Lagrange multiplier of the non-negativity constraint $K_i \geq 0$. Invoking [12], gradient and expectation can be interchanged to yield $\mathrm{E}\lambda_i(K^{solo}, D) = c_i - \nu_i^{solo}$,where λ_i is firm's i capacity shadow value: $\lambda_i = \frac{\partial \pi_i}{\partial K_i}$. The shadow value λ_i, which is the optimal dual variable of firm i's production linear program, equals a constant λ_i^l in each domain Ω_l of Figure 2. Thus, the expected marginal revenue can be expressed as $\frac{\partial \pi_i}{\partial K_i} = \mathrm{E}\lambda_i = \sum_{l=1}^{6} \lambda_i^l P(\Omega_l(K))$. To simplify notation, define a 2×6 matrix Λ whose l-th column is the shadow vector in domain $\Omega_l : \Lambda_{il} = \lambda_i^l$. Similarly, define a 6×1 vector $\bar{P}(K)$ whose l-th coordinate is the probability of domain $\Omega_l : \bar{P}_l(K) = P(\Omega_l(K))$. When both firms "go solo" the marginal vector is

$$\mathrm{E}\lambda = \Lambda^{solo} \bar{P}(K^{solo}) = \begin{bmatrix} p_M & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & 0 & 0 \end{bmatrix} \bar{P}(K^{solo}) = \begin{bmatrix} p_M P(D_1 > K_1^{solo}) \\ p_S P(D_2 > K_2^{solo}) \end{bmatrix}.$$

Because contribution margins exceed investment costs ($p_i > c_i$) both firms will invest ($\nu^{solo} = 0$) and the optimality equations directly yield the simple newsvendor solutions $G_i\left(\frac{c_i}{p_i}\right)$.

2.3 Centralization

When both firms are controlled by one central decision maker, the optimal production and sales vector x in stage two maximizes system revenue, subject to system capacity and demand constraints. Transfers x_t are possible and optimal activity levels x^{cen} and revenue π^{cen} are the solution of the product mix linear program:

$$\begin{aligned} \pi^{cen} &= \max_{x \geq 0} p_M(x_M + x_t) + p_S x_S \\ \text{s.t. } x_M &\leq K_M, x_t + x_S \leq K_S, x_t + x_M \leq D_M, x_S \leq D_S. \end{aligned} \tag{2}$$

The optimal investment vector K^{cen} maximizes expected system value:

$$K^{cen} = \arg\max_{K \geq 0} V^{cen}(K) \qquad \text{where} \qquad V^{cen}(K) = \mathrm{E}\pi^{cen}(K, D) - c'K. \tag{3}$$

The option of transfers x_t enlarges the supply system's capacity region to $\Omega_0 \cup \Omega_1$, or Ω_{01} in short. Using this shorthand notation, if $D \in \Omega_{23456}$, demand exceeds supply and the optimal supply vector $X = (x_M + x_t, x_S)$ will be on the boundary of the capacity region Ω_{01}. The linear program (2) can be solved parametrically in terms of K and D (thereby directly manifesting the domains Ω_l defined earlier). If market M is more profitable than market S, it gets priority in the capacity allocation decision yielding market supply vector X_b in Figure 2. Otherwise market S gets priority yielding vector X_a in Figure 2. As before, $\pi^{cen}(K, D)$ is concave and the shadow vector $\lambda(K, D)$ is constant in each domain so that the optimal capacity vector K^{cen} solves $\Lambda^{cen}\bar{P}(K^{cen}) = c - \nu^{cen}$ and $K^{cen\prime}\nu^{cen} = 0$, where

$$\Lambda^{cen} = \begin{bmatrix} 0 & 0 & \min(p) & p_M & p_M & \min(p) \\ 0 & p_S & p_S & \max(p) & p_M & \min(p) \end{bmatrix}. \tag{4}$$

If M-capacity is less expensive than S-capacity ($c_M < c_S$), it is profitable to invest in both types of capacity ($\nu^{cen} = 0$). Otherwise, it is optimal to supply both markets using only the cheaper S-capacity: $\nu_M^{cen} > 0$ and $K_M^{cen} = 0$. In the Appendix of [23] we show that V^{cen} is strict concave at K^{cen} so that the optimal investment vector is unique.

We now have completely characterized the optimal investment strategies in both reference scenarios. Clearly, system values under centralization V^{cen} (weakly) dominate those when both players go solo: $V^{cen} \geq V_+^{solo} = V_1^{solo} + V_2^{solo}$. The value gap $\Delta V^{solo} = V^{cen} - V_+^{solo}$ captures the costs of decentralization. In the remainder of this article, we will investigate how subcontracting can decrease the value gap and whether it can "coordinate" the supply network. That is, can subcontracting increase system efficiency and eliminate the value gap?

3 Subcontracting with Price-Only Contracts

A *price-only contract* specifies ex-ante the transfer (or "wholesale") price p_t that the manufacturer must pay for each unit supplied by the subcontractor. Because this simple contract does not specify a transfer quantity x_t or any other model variables, it cannot force a party to enter the subcontracting relationship. Using Cachon and Lariviere's [6] terminology, contract compliance is voluntary and both parties will enter the subcontracting relationship (or "trade") only if it benefits them. First we will consider p_t as given and known by both parties from the start and analyze this contract structure for any value of p_t. Later we will discuss the choice—or contract design—of the transfer price p_t.

As before, both players must decide separately, yet simultaneously, on their capacity investments in stage 1 before uncertainty is resolved. The resulting capacity vector K is observable and becomes common information. After demand is realized, both parties make their individual production-sales decisions x in stage 2 where they have the option to subcontract. Thus, the manufacturer can ask the subcontractor a supply x_t^M and the subcontractor has the option to fill the order x_t^M (up to her capacity constraint). Thus the subcontractor offers a quantity $x_t^S \leq x_t^M$, which is accepted by the manufacturer in exchange for a payment $p_t x_t$.

When making decisions, each player acts strategically and takes into account the other player's decisions. Any capacity vector K (production vector x) with the property that no player can increase firm value by deviating unilaterally from K (x) is a Nash equilibrium in pure strategies and is called simply an *optimal investment* (production) *vector*. Its resulting firm value (revenue) vector is denoted by $V(K)$ ($\pi(x)$). The analysis of our subcontracting model involves establishing and characterizing the existence of a Nash equilibrium in this two-player, two-stage stochastic game. As with any dynamic decision model, we start with stage 2 and solve the *production-subcontracting subgame* for any given pair (K, D). We will show that there exists a unique optimal revenue vector $\pi(K, D)$, which will allow us to solve the full *investment game* in stage 1.

3.1 The Production-Subcontracting Subgame

For any given capacity vector $K \geq 0$, both players decide sequentially on their production and transfer levels in order to maximize their own revenue:

$$\begin{array}{rrcl} \max\limits_{x_M, x_t, x_t^M \geq 0} & p_M x_M & + & (p_M - p_t) x_t \\ \text{s.t.} & x_M & \leq & K_M, \\ & x_M + x_t & \leq & D_M, \\ & x_t & = & \min(x_t^M, x_t^S), \end{array} \quad \text{and} \quad \begin{array}{rrcl} \max\limits_{x_S, x_t^S \geq 0} & p_S x_S & + & p_t x_t \\ \text{s.t.} & x_S + x_t^S & \leq & K_S, \\ & x_S & \leq & D_S, \\ & x_t & = & \min(x_t^M, x_t^S). \end{array}$$

Depending on the value of p_t, the manufacturer M and supplier S have a higher or lower incentive to subcontract. First, M will only subcontract if $p_t < p_M$, otherwise the independent solo solution emerges. Thus, for the remaining of this article we will assume $p_t < p_M$ so that M will always prioritize his internal capacity and will ask S to fill the remaining demand: $x_M = \min(D_M, K_M)$ and $x_t^M = D_M - x_M = (D_M - K_M)^+$. Second, S has an incentive to fill M's demand if $p_t > p_S$, while if $p_t < p_S$, she will prefer to fill her own market demand. Thus, we must distinguish between two cases: high transfer price ($p_S < p_t < p_M$) and low transfer price ($p_t \leq \min(p)$).

If the transfer price is high, S prefers supplying M to serving her own market and will fill M's order to the best of her capacity: $x_t^S = \min(K_S, x_t^M)$. Thus, the subcontracting supply is $x_t = \min\left((D_M - K_M)^+, K_S\right)$, and both players prefer subcontracting whenever M has excess demand, that is if $D \in \Omega_{13456}$. S will use any remaining capacity to fill her own market demand: $x_S = \min(D_S, K_S - x_t)$. (The resulting market supply vector in Figure 2 is X_b.) If the transfer price is low, S has little incentive to supply M and prefers serving her own market: $x_S = \min(D_S, K_S)$. S will use any remaining capacity to fill M's demand: $x_t^S = \min(x_t^M, K_S - x_S)$. Thus the subcontracting supply is $x_t = \min\left((D_M - K_M)^+, (K_S - D_S)^+\right)$, and subcontracting will materialize when M has excess market demand *and* S has low market demand, that is if $D \in \Omega_{156}$. (The resulting market supply vector in Figure 2 is X_a.)

In both cases (high or low transfer price), the production vector x forms a unique Nash equilibrium because no player has an incentive to deviate unilaterally. At any transition point between the regimes where an equality sign holds (e.g., $p_S = p_t$), players are indifferent between the two regimes because they receives the same revenue in either regime, and a continuum of production vectors are Nash equilibria. This poses no problems, however, because linear programming theory yields that the associated revenue vector $\pi(K, D)$ is unique and concave in K:

Proposition 1 *For any demand D and capacity K vector, there is at least one Nash equilibrium $x(K, D)$ in pure strategies and all equilibria have identical revenue vector $\pi(K, D)$, which is concave in K.*

3.2 The Full Capacity Investment Game

To demonstrate the existence of a Nash equilibrium in pure strategies, we will show that the *capacity reaction curves* have an intersection point that is stable. (A simple three step sequential game like our subgame always has an equilibrium.) Firm i's capacity reaction curve $k_i(\cdot)$ specifies firm i's optimal investment level $K_i = k_i(K_j)$ given that firm j has capacity K_j. Thus, $k_i(\cdot)$ is defined pointwise for each $K_j \geq 0$ as $k_i(K_j) = \arg\max_{K_i \geq 0} V_i(K)$. As before, because $\pi_i(K, D)$ is concave in K_i, so too is the linear superposition $\mathrm{E}\pi_i(K, D)$ and thus $V_i(\cdot)$. Thus, the first order conditions (FOC) are sufficient and can be represented in matrix notation:

$$\Lambda^{sub}\bar{P}(K) = c - \nu \text{ and } \nu' K = 0, \tag{5}$$

where

$$\Lambda^{sub} = \begin{cases} \begin{bmatrix} p_t & 0 & p_t & p_M & p_M & p_t \\ 0 & p_S & p_S & p_t & p_t & p_S \end{bmatrix} & \text{if } \; p_S \leq p_t < p_M, \\ \\ \begin{bmatrix} p_t & 0 & p_M & p_M & p_M & p_M \\ 0 & p_S & p_S & p_S & p_t & p_t \end{bmatrix} & \text{if } \; p_t < \min(p_S, p_M). \end{cases}$$

Thus, firm $i's$ reaction curve is found by solving equation i in (5) as a function of K_j. Implicit differentiation of the FOC shows that $-1 \leq \frac{dk_i}{dK_j} \leq 0$ (details can be found in the Appendix of [23]). Axis crossings and asymptotes are as shown in Figure 3. It directly follows that the reaction curves have an intersection K^{sub}. Moreover, at least one reaction curve has a slope $\frac{dk_i}{dK_j} > -1$ at an intersection so that the corresponding equilibrium is unique and stable (Nash).

Proposition 2 *The unique solution K^{sub} of (5) is the unique optimal investment vector.*

3.3 Complete Subcontracting (Outsourcing): $K_M^{sub} = 0$

From the structure of the capacity reaction curves, it follows that the optimal investment strategy has one of two distinct forms. Either both firms invest or only the supplier invests. In the latter case, the manufacturer relies for all sourcing on the outside party. Formally, one can express an *outsourcing condition* in terms of a threshold $\bar{c}_M$ on the manufacturer's investment cost c_M as follows. Set $\bar{K} = (0, k_S(0))$ and define the threshold cost $\bar{c}_M = \Lambda_{1,:}^{sub}\bar{P}(\bar{K})$, where $\Lambda_{1,:}^{sub}$ is the first row of Λ^{sub}. Then the manufacturer should outsource if and only if his investment cost c_M exceeds the threshold cost $\bar{c}_M$.

Clearly, the threshold cost $\bar{c}_M$ depends on the manufacturer's margin p_M, on "his cost to subcontract" as expressed by the transfer price p_t, and on the joint probability measure P of the demand forecast. The supplier's margin p_S and investment cost c_S, however, also impact the outsourcing decision, reflecting the strategic interactions in our game-theoretic model. In the Appendix of [23] we show that for low levels of demand uncertainty, the threshold level is *independent* of the demand distribution:

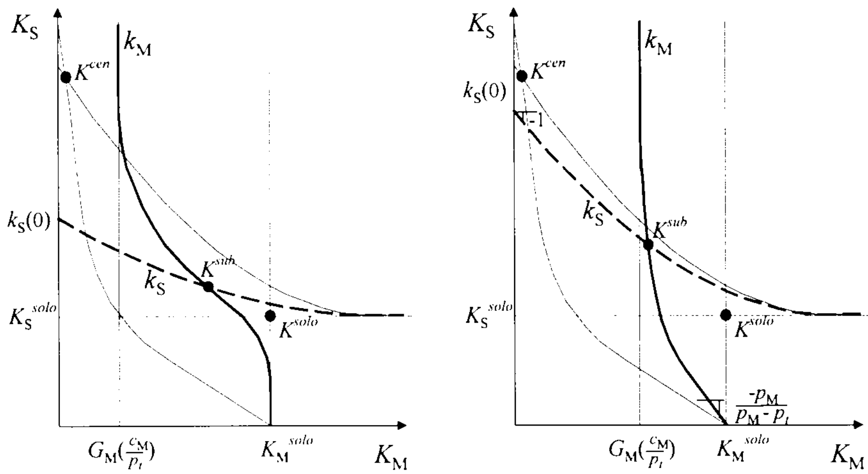

Figure 3: The intersection of the capacity reaction curves k_M (bold) and k_S (dashed bold) defines the optimal investment K^{sub} when subcontracting with a price-only contract with low p_t (left) and high p_t (right).

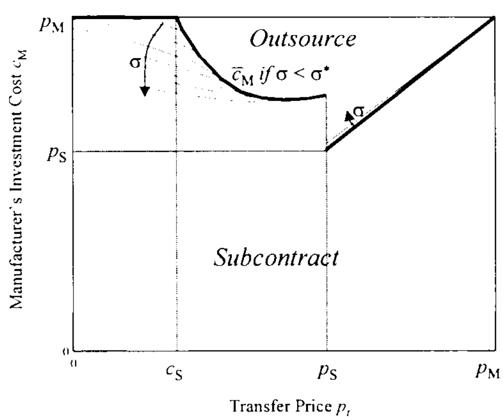

Figure 4: The threshold cost $\bar{c}_M$ (shown for increasing levels of uncertainty as measured by the standard deviation σ of demand assuming $\sigma_{D_M} = \sigma_{D_S}$) partitions the strategy space.

$$\bar{c}_M = \begin{cases} p_M & \text{if} \quad p_t < c_S, \\ p_t + \left(\frac{p_M}{p_t} - 1\right) c_S & \text{if} \quad c_S \leq p_t < p_S, \\ p_t & \text{if} \quad p_S \leq p_t < p_M. \end{cases} \tag{6}$$

Thus, with little demand uncertainty and low transfer prices, no outsourcing will happen because such low transfer prices give the supplier not enough incentive to invest in extra capacity to serve the manufacturer. (Even when $p_t < \min(c)$, M must still invest in in-house capacity because supply is not guaranteed by S who will prioritize her own market when capacity-constrained.) For transfer prices higher than the supplier's capacity cost, outsourcing is possible. For medium transfer prices, the threshold $\bar{c}_M$ is decreasing in p_t so that outsourcing becomes more likely with higher transfer prices p_t. When the transfer price exceeds the suppliers margin, a discontinuous drop in $\bar{c}_M$ reflects the fact that the supplier now has a very strong incentive to invest in extra capacity. As the transfer price increases, however, subcontracting increasingly becomes more expensive for the manufacturer compared to in-house capacity so that outsourcing becomes less likely. (From

	c_M	c_S	p_M	p_S	p_t
K_M^{sub}	$-(\alpha_1+\alpha_3)$	α_2	$(\alpha_1+\alpha_3)P_{45}$	$-\alpha_2 P_{236}$	$(\alpha_1+\alpha_3)P_{136}-\alpha_2 P_{45}$
K_S^{sub}	α_1	$-(\alpha_2+\alpha_4)$	$-\alpha_1 P_{45}$	$(\alpha_2+\alpha_4)P_{236}$	$-\alpha_1 P_{136}+(\alpha_2+\alpha_4)P_{45}$
V_M^{sub}	$\beta_1-K_M^{sub}$	$-\beta_2$	$Ex_{M+t}^{sub}-\beta_1 P_{45}$	$\beta_2 P_{236}$	$\beta_5\frac{\partial K_S^{sub}}{\partial p_t}-c_M\frac{\partial K_M}{\partial p_t}-Ex_t^{sub}$
V_S^{sub}	β_3	$-\beta_4-K_S^{sub}$	$-\beta_3 P_{45}$	$Ex_S^{sub}+\beta_4 P_{236}$	$-\beta_6\frac{\partial K_M^{sub}}{\partial p_t}-c_S\frac{\partial K_S}{\partial p_t}+Ex_t^{sub}$

Table 1: Sensitivity of the optimal investment levels K^{sub} and value V^{sub}, where $\alpha,\beta \geq 0$. Table entries represent partial derivatives: $\alpha_1 = \frac{\partial K_S^{sub}}{\partial c_M}$ for example.

the structure of the manufacturer's reaction curve it follows that the threshold cost $\bar{c}_M$ cannot be smaller than p_t, because a necessary condition for outsourcing is that $c_M \geq p_t$ so that $G_M(\frac{c_M}{p_t}) = 0$.) Figure 4 illustrates the outsourcing vs. (partial) subcontracting strategies. Recall that a centralized system would not invest in manufacturing capacity (and hence "outsource") if $c_M > c_S$, covering a wider 'outsourcing' zone of the strategy space. This is a first indication that subcontracting with simple price contracts will improve system coordination as compared to the solo scenario (never outsourcing), yet it will not eliminate the value gap ΔV.

When the level of uncertainty in the demand forecast rises above a certain level, the threshold cost $\bar{c}_M$ will decrease for low to medium transfer prices ($p_t < p_S$) but increase for high transfer prices ($p_S < p_t < p_M$). Thus, for low to medium transfer prices, more uncertainty creates a stronger incentive for the supplier to invest in extra capacity making outsourcing more likely. For high transfer prices, on the other hand, more uncertainty increases the expected total transfer cost to the manufacturer who will prefer more in-house capacity making outsourcing less likely.

3.4 Sensitivity of the Investment-Subcontracting Strategies

The sensitivity of the optimal investment strategy with respect to changes in capacity costs c, contribution margins p, and transfer price p_t is summarized in Table 1. Let us highlight some interesting factors. First, strategic decision making captured by our game-theoretic model makes one party's investment level and firm value dependent on the other party's cost and revenue structure. When the manufacturer faces higher investment costs, for example, he will decrease his investment level. The supplier, on the other hand, anticipates the manufacturer's decisions and her decision reflects the externalities in our model. Lower manufacturing capacity most likely will lead to higher supply requests x_t^M, giving the supplier an incentive to increase her investment level. The increase in K_S^{sub}, however, does not make up for the decrease in K_M^{sub} (because transfers are only made with a probability strictly less than one). This shows that *optimal manufacturing and supplier capacity levels are imperfect substitutes* with respect to capacity costs c and margins p.

Second, *optimal manufacturing and supplier capacity levels are more sensitive to changes in capacity costs c than changes in output prices (margins) p.* This is a direct result from the presence of uncertainty. For example, an increase in p_M only warrants an increase in manufacturing capacity if demand is sufficiently large (e.g., $D \in \Omega_{45}$ if $p_t > p_S$). An increase in c_M, on the other hand, always justifies a decrease in manufacturing capacity, regardless of the demand realization. This result is in stark contrast to deterministic systems and one expects this sensitivity differential to increase in the amount of demand variability.

Third, while the supplier's value sensitivity directly reflects the externalities in the model, the manufacturer's value is a little more intricate. Clearly, an increase in supplier costs leads to a decrease in total system capacity, which impacts both parties' value negatively. An increase in manufacturing cost benefits the supplier who increases her capacity in anticipation of a larger total demand $x_t^M + D_S$. This effect can dominate to yield the unexpected result that the manufacturer's

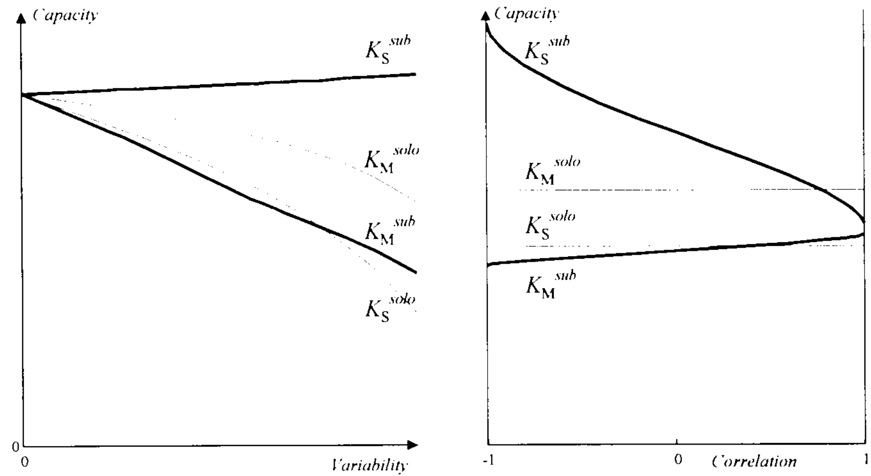

Figure 5: Optimal investment levels as a function of demand variability when market demands are uncorrelated (left) and of correlation when market demands are uniform (right).

value *can be increasing in its investment cost.* The manufacturer enjoys spill-over benefits from increased supplier capacity that may outweigh his increased investment costs.

Fourth, the table shows that an increase in the transfer price p_t has a similar effect as a *simultaneous increase in margins* p_M and p_S. The absolute effect on investment levels and firm values is ambiguous. An increase in p_t makes subcontracting more expensive for the manufacturer relative to internal capacity investment. This is reflected by a rightward move of the manufacturer's reaction curve k_M in Figure 3. Increased transfer prices, however, give the supplier a higher incentive to increase her "relationship-specific" investment. Thus, while we expect K_M^{sub} to decrease and K_S^{sub} to increase, the supplier's reaction curve k_S can move upward more than k_M moves right so that K_M^{sub} increases and K_S^{sub} decreases; again, illustrating the intricate externalities that can occur in stochastic games.

Finally, to study the effect of uncertainty on the optimal investment strategies, we consider a probability measure $P(\cdot\,|\,\gamma)$ with density $f(\cdot\,|\,\gamma)$ that is parameterized by γ, where γ represents an uncertainty measure of importance such as variability or correlation. Formally, the impact of changes in γ on the optimal investment strategy can be expressed as:

$$\frac{\partial}{\partial\gamma}K^{sub} = -|J|^{-1}\begin{bmatrix} \sum_{l=1}^{6}\left(J_{22}\Lambda_{1l}^{sub} - J_{21}\Lambda_{2l}^{sub}\right)P_l^{\gamma} \\ \sum_{l=1}^{6}\left(-J_{12}\Lambda_{1l}^{sub} + J_{11}\Lambda_{2l}^{sub}\right)P_l^{\gamma} \end{bmatrix},$$

where J is the Jacobian of the optimality equations (5) and

$$P_l^{\gamma} = \frac{\partial}{\partial\gamma}P(\Omega_l(K^{sub})\,|\,\gamma) = \int_{\Omega_l(K^{sub})}\frac{\partial}{\partial\gamma}f(z\,|\,\gamma)dz.$$

Although this expression is of limited practical value, it may be useful for estimating the sign of $\frac{\partial}{\partial\gamma}K^{sub}$. The appendix of [23] shows that $J_{22} \le J_{21} \le 0$ and $J_{11} \le J_{12} \le 0$. Thus, $\frac{\partial}{\partial\gamma}K_M^{sub}$ and $\frac{\partial}{\partial\gamma}K_S^{sub}$ may have *opposite signs* so that the optimal manufacturer and supplier investment levels would respond in opposite ways to changes in the demand distribution, akin to the substitution effect stated earlier. This effect is present for changes in the level of demand uncertainty or demand correlation in the example shown in Figure 5. This example was generated numerically using a demand distribution parameterized by correlation and standard deviation in market demand. Given that the mean is constant, we can use the standard deviation as a measure of variability or market risk. Correlation varies between -1 and $+1$ for perfect negatively and positively correlated demand, respectively. (Explicit expressions for this family of distributions were first presented in [22, pp. 75-77].) For simplicity we assume identical mean and standard deviations for D_M and D_S, so that both markets are 'equally risky'.

As shown in the left graph of Figure 5, optimal investment levels are monotone in variability but they can be increasing or decreasing. This is similar to the well-known effect in one-dimensional newsvendor models with symmetric demand distributions where optimal investment increases (decreases) in variability if the critical ratio $\frac{c}{p} > 0.5$ (< 0.5). Also, the supplier's investment increases when the manufacturer's investment decreases, and vice versa. More importantly, compared to the independent "solo" setting, an increase in market risk *decreases* the manufacturer's relative investment if there is a subcontracting option. This can be paraphrased as saying that *the manufacturer will subcontract more as market risk increases* and the subcontractor's response is to invest more. (The subcontractor's optimal investment level seems to be less sensitive to risk, which may be explained by risk pooling: the supplier's effective demand pools over both markets and therefore is less variable.) The graph at the right in Figure 5 shows that the manufacturer's (supplier's) investment level is increasing (decreasing) in the correlation between the two market demands. Thus, *the manufacturer will subcontract less as market correlation increases.* Indeed, when market demands are positively correlated the subcontracting option has less value so that the optimal fraction of capacity that is subcontracted decreases.

3.5 System Coordination and the Value Gap $\Delta V = V^{cen} - V_+^{sub}$

Comparing the capacity reaction curves (in bold in Figure 3) with the optimality curves that define the optimal centralized and solo investment (in light in Figure 3) directly yields:

$$K_M^{cen} \leq K_M^{sub} \leq K_M^{solo} \qquad \text{and} \qquad K_S^{cen} \geq K_S^{sub} \geq K_S^{solo}.$$

Intuitively, this is what one expects: subcontracting allows the manufacturer to decrease his investment in capital and/or labor. The option of subcontracting means potentially more business for the supplier and thus warrants additional (or 'relationship-specific') investment. Figure 6 illustrates how the subcontracting capacity investment levels compare to those in the reference scenarios as a function of the transfer price p_t. As argued earlier, the capacity levels are imperfect substitutes while total industry investment level K_+^{sub} is increasing in p_t. The figure also shows that in the context of our model *subcontracting may reduce or increase industry investment* compared to the solo or centralized setting. (While the figure shows that $K_+^{solo} < K_+^{sub}$, this is not true in general either.) Thus, in contrast with the one-dimensional competitive models of Li [17] and Lippman and McCardle [18], in our model centralization and its implicit monopoly power need not result in industry under-investment compared to the subcontracting and independent (solo) settings. Similarly, subcontracting and outsourcing need not result in a decrease of total capacity compared to the solo setting.

A key question is how effective price-only subcontracting is in inducing system coordination; that is, by how much does it reduce the costs of decentralization as measured by the value gap $\Delta V = V^{cen} - V_+^{sub}$? Because the optimal centralized and subcontracting investment vectors are the unique solutions to $\Lambda^{cen}\bar{P}(K^{cen}) = c - \nu^{cen}$ and $\Lambda^{sub}\bar{P}(K^{sub}) = c - \nu^{sub}$, respectively, and $\Lambda^{cen} \neq \Lambda^{sub}$, both investment vectors are different in general: $K^{cen} \neq K^{sub}$ and thus system value $V_+^{sub} < V^{cen}$ (because the value functions are strictly concave at the optimal investment vectors). Therefore, *subcontracting with price-only contracts does not coordinate the supply network.* It does, however, mitigate the decentralization costs in that it reduces the value gap compared

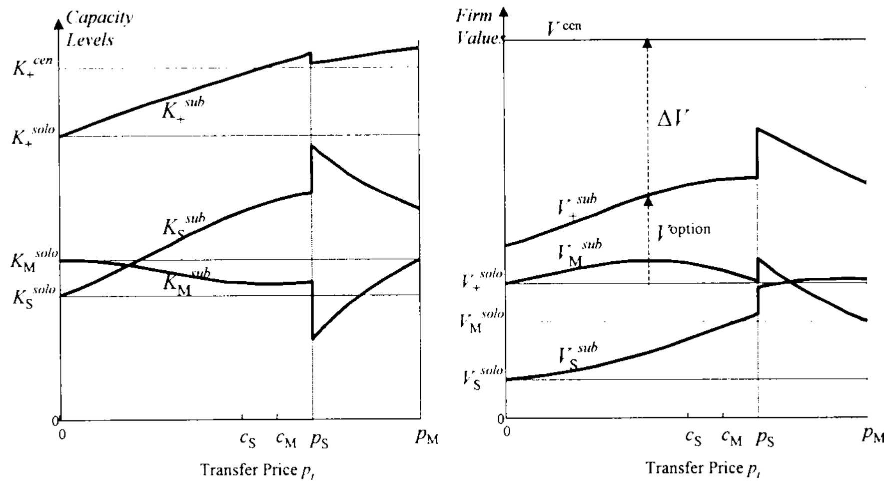

Figure 6: Capacity levels, the option value of subcontracting V^{option} and the decreased value gap ΔV as a function of the transfer price p_t when market demands are uniform but strongly negatively correlated.

to the no-subcontracting setting. From the structure of the capacity reaction curves shown in Figure 3 it is clear that the optimal subcontracting investment vector K^{sub} moves toward the centralized investment K^{cen} (and thus V_+^{sub} moves toward V^{cen}) if c_M increases, *ceteris paribus* (so that k_S remains unchanged). In that case, an increasing supplier cost advantage improves system coordination compared to the solo setting. Indeed, subcontracting becomes more profitable to both parties when the supplier has a cost advantage and her capacity increase can be made at lower cost than if the manufacturer were to invest himself.

The impact of the contract transfer price p_t, however, is ambiguous. Because both reaction curves change as p_t changes, the result on K^{sub} and thus V_+^{sub} is unclear. Higher transfer prices give higher incentives to the supplier yet lower to the manufacturer. The overall result on firm and industry values can go either way because of the externality effects in our strategic model. (Partial p_t-derivatives in Table 1 cannot be signed in general.) *Contract design*, or the choice of the optimal p_t (whether one wants to maximize manufacturer, supplier or system profits–depending on which party has most 'power' in setting p_t), becomes thus very case specific. In all our numerical test problems, system profits where maximized at $p_t = p_S$ yielding a substantial improvement in the value gap ΔV, which is in agreement with economic theory stating that transfer prices should be set equal to outside opportunity costs. If the manufacturer sets the transfer price, however, he does not necessarily set it at p_S. Indeed, because of demand variability, a transfer price below p_S may yield optimal profits for the manufacturer. Figure 6 illustrates this possibility when market demands are strongly negatively correlated ($\rho = -0.9$).

Finally, the presence of demand uncertainty is a key driver in the option value of subcontracting. Figure 7 illustrates that the option value of subcontracting is increasing in variability. Thus, similar to many financial options, more uncertainty is good for this real option. In absolute terms, however, more variability reduces firm values. Figure 7 also confirms intuition that negative demand correlations increase the option value of subcontracting. In terms of our graphical solution technique of Figure 2, the triangular option region Ω_1 gets more probability mass as correlation becomes more negative.

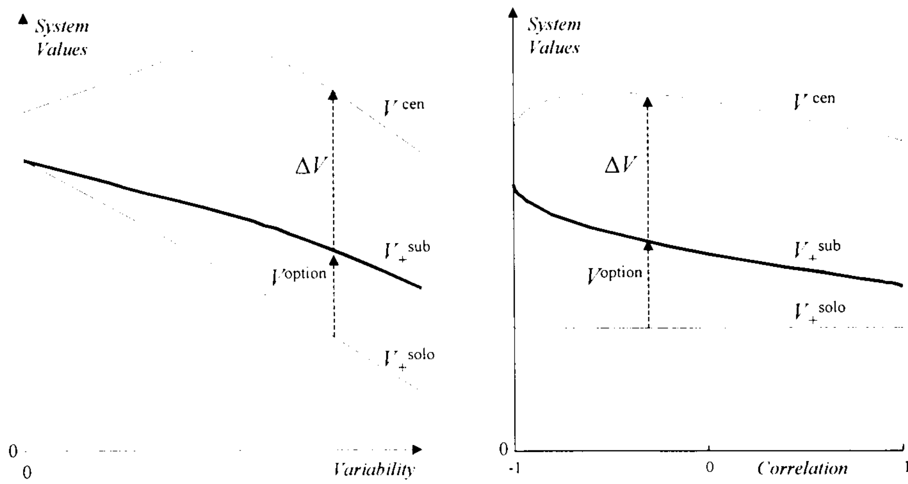

Figure 7: The option value of subcontracting V^{option} and the decreased value gap ΔV as a function of demand variability when market demands are uncorrelated (left) and of correlation when market demands are uniform (right).

4 Subcontracting with Other Contracts

Many contract structures other than price-only contracts can be used to regulate subcontracting. In this section we will discuss two other types of contracts and relate our results to more complex contracts studied in the literature.

4.1 Incomplete Contracts: Bargaining

In some situations, ex-ante contracts may be too expensive or impossible to specify or enforce. Start-up companies and companies in developing countries may find it too expensive to enforce execution of a contract [11], while "investments by suppliers in quality, information sharing systems, responsiveness and innovation are often non-contractible. Without the ability to specify contractually in advance the division of surplus from non-contractible investments, this surplus will be divided based on the ex-post bargaining power of the parties involved [3]." This *incomplete contracts approach* was first suggested by Grossman, Hart and Moore [9, 13] to study vertical integration. In our setting, it may be thought of as the ultimate minimalistic and opportunistic approach to subcontracting: no contracts are needed and subcontracting only happens if both parties profit from it.

The model is similar to before and both firms have the option to engage in a trade at the beginning of stage two. The firms can decide jointly on production-sales decisions so that the resulting activity vector equals the vector $x^{cen}(K, D)$ chosen in the centralized scenario. Engaging in subcontracting thus yields a revenue surplus $\Delta\pi(K, D) = \pi^{cen}(K, D) - \pi_+^{solo}(K, D) \geq 0$ compared to going solo, and both parties thus have an incentive to engage in the trade $x_t(K, D)$. Without the ability to contractually specify in advance the division of the surplus, the firms must negotiate this division, which can be cast as bilateral bargaining. Many bargaining games are possible (c.f. Kamien and Li [14, p. 1357]). Nash introduced a game that leads to splitting the surplus evenly. Rubinstein presents a sequential game in which player i gets fraction $\theta = \frac{1-\delta_i}{1-\delta_i\delta_j}$ of the surplus, where δ_i is the "impatience" or discount factor of player i, which is ex-ante observable. Whichever bilateral bargaining game is used, the manufacturer can ex-ante *expect* (but not contractually specify) to receive fraction θ of the surplus while the supplier will get fraction $\bar{\theta} = 1 - \theta$. One can

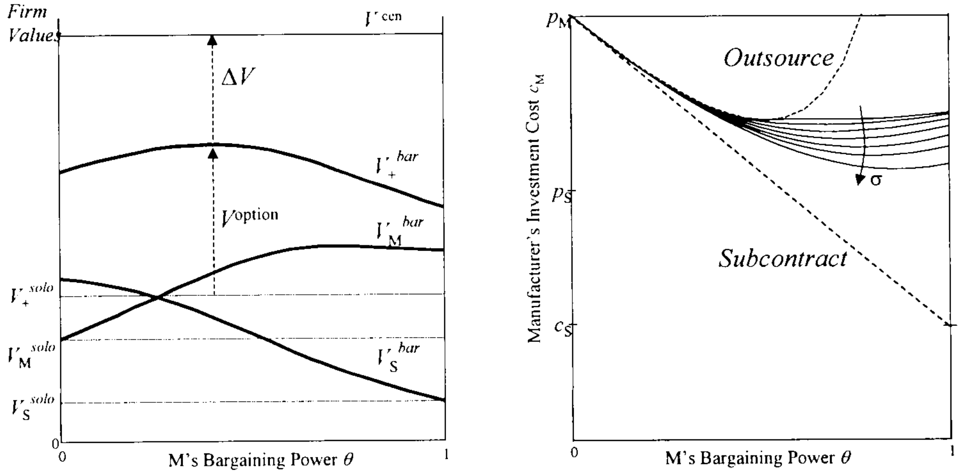

Figure 8: The option value of subcontracting with an incomplete contract and its outsourcing threshold (with dashed bounds) as a function of the manufacturer's bargaining power for the same model parameters as Figure 7.

also think of θ as the 'bargaining power' of the manufacturer.

In summary, with incomplete contracting the manufacturer's revenue function is $\pi_M^{solo} + \theta\pi^{cen}$ while the supplier receives $\pi_S^{solo} + \bar{\theta}\pi^{cen}$. Thus, as before, the capacity reaction curves can be constructed in terms of a shadow matrix $\Lambda^{bar} = \Lambda^{solo} + diag(\theta, \bar{\theta})\Lambda^{cen}$. Both curves have a unique, stable intersection that defines the optimal investment vector K^{bar}. Thus, whereas price-only contracts can explicitly induce additional supplier investment through a high transfer price, the division of the ex-post surplus here gives the supplier an indirect incentive to make a relationship-specific investment.

Because the sensitivity of the investment strategy is similar to that under price-only contracts, we will focus on the role of the bargaining power θ. As earlier, we can express an outsourcing condition in terms of a threshold $\bar{c}_M$ on the manufacturer's investment cost c_M. The appendix of [23] shows that if the supplier has higher margins ($p_S > p_M$), outsourcing will never be optimal ($\bar{c}_M > p_M$). If, however, the manufacturer has higher margins ($p_M \geq p_S$), outsourcing is possible as shown by the bounds on the outsourcing threshold:

$$c_M < \theta c_S + \bar{\theta} p_M \leq \bar{c}_M \leq \min\left(p_M, \bar{\theta} p_M + \frac{\theta}{\bar{\theta}} c_S\right).$$

The threshold is decreasing (almost linearly) for small θ which implies that *outsourcing is more likely for more powerful manufacturers.* The argument, however, cannot be generalized to very powerful contractors ($\theta \to 1$): the threshold may be *increasing* close to $\theta = 1$ as shown in Figure 8. There seems to be a range of bargaining powers around $\theta_b \simeq 0.75$ (and decreasing in demand variability) for which outsourcing is most likely. If the contractor's bargaining power is substantially higher, outsourcing is less likely because the subcontractor receives less ex-post surplus and has less ex-ante incentive to make a relation-specific investment. If bargaining power is much smaller, most surplus goes to the supplier. As shown in Figure 8, *system* value and the option value of subcontracting with incomplete contracts is maximal when surplus is divided not too unevenly (but it need not be a fair $50-50$ split). More importantly, incomplete contracts are not inferior to explicit price-only contracts. For example, comparing Figure 8 with corresponding Figures 6 and 4 shows that the option value can be larger and that outsourcing is more likely. Yet, as before, mere supplier cost advantage of the subcontractor is *not* sufficient for the manufacturer to outsource.

Indeed, even if $c_M > c_S$, as long as $c_M < \bar{c}_M$ it is optimal for the manufacturer to invest in some in-house capacity. Thus, neither price-only nor incomplete contracts can coordinate this supply system.

4.2 State-dependent Price-Only Contracts

A *state-dependent price-only contract* is a price-only contract that specifies an ex-ante transfer price for each possible state vector. The price can be demand dependent $p_t(D)$, or also dependent on the capacity vector $p_t(K, D)$. (The latter assumes that capacity levels are not only observable by the two firms as assumed earlier, but also verifiable by a third party.) Obviously, because of the increased degrees of freedom, optimal contract design of a state-dependent contract (a calculus of variation problem) will improve performance (system or one player's depending on the objective) compared to a constant price-only contract. Nevertheless, *even state-dependent price-only contracts do not coordinate the supply system* in general. Indeed, coordination would require that the expected marginal revenues equal those in the centralized scenario. Because

$$\mathrm{E}\lambda^{\mathrm{sub}\ p_t} = \begin{bmatrix} p_M P_{45} + \int_{\Omega_1} p_t dP + \int_{\Omega_{36}} \left[p_t 1_{\{p_t \geq p_S\}} + p_M 1_{\{p_t < p_S\}} \right] dP \\ p_S P_{23} + \int_{\Omega_4} \max(p_t, p_S) dP + \int_{\Omega_5} p_t dP + \int_{\Omega_6} \min(p_t, p_S) dP \end{bmatrix},$$

it directly follows that $\mathrm{E}\lambda_M^{\mathrm{sub}\ p_t} \geq \mathrm{E}\lambda_M^{cen}$ and $\mathrm{E}\lambda_S^{\mathrm{sub}\ p_t} \leq \mathrm{E}\lambda_S^{cen}$. Coordination requires equality or $P_{13456} = 0$ (and thus outsourcing and high c_M) if $p_S \leq p_M$, or $P_{156} = 0$ (and thus $K^{\mathrm{sub}\ p_t} = K^{solo}$ if P has (unusual) non-convex support) if $p_S > p_M$. Thus, in stochastic systems with partial subcontracting coordination is generally not achieved with our three types of simple contracts. Not surprisingly, the higher complexity of subcontracting makes coordination more difficult compared to traditional outsourcing models in supply chains.

This contract type also allows us to relate the price-only contract with the bargaining contract. Indeed, the execution of the inter-firm supply $x_t^{bar}(K^{bar}, D)$ and the division of the surplus is implemented by specifying the quantity $x_t(K, D)$ to be provided by the subcontractor and the transfer price p_t^{bar} to be paid by the manufacturer for each unit provided. This transfer price is defined implicitly in the bargaining model in that it guarantees the correct division of surplus: $\pi_S^{bar} = p_S x_S^{cen} + p_t^{bar} x_t^{cen}$ (recall that $x^{bar} = x^{cen}$). Rearranging terms yields

$$p_t^{bar} x_t^{cen} = \bar{\theta} p_M x_t^{cen} + \theta p_S (x_S^{solo} - x_S^{cen}), \tag{7}$$

and because $x_t^{cen} \geq x_S^{solo} - x_S^{cen}$ we have that $\bar{\theta} p_M \leq p_t^{bar}(K, D) \leq \bar{\theta} p_M + \theta p_S$.The payment $p_t^{bar} x_t^{cen}$ is the composition of two terms: $p_M x_t^{cen}$ is the gross surplus derived from subcontracting while $p_S(x_S^{solo} - x_S^{cen})$ is the subcontractor's opportunity cost or the profit forgone by subcontracting. The gross surplus is received by the manufacturer who pays the share $\bar{\theta} p_M x_t^{cen}$ to the subcontractor. The subcontractor bears the opportunity cost and is compensated by the contractor for the share $\theta p_S(x_S^{solo} - x_S^{cen})$.

Moreover, if the manufacturer has a margin advantage but limited bargaining power such that $\theta p_M < p_M - p_S$, then a price-only contract with state-dependent transfer price $p_t = p_t^{bar}(K, D)$ will yield the same investment vector as an incomplete bargaining contract. Indeed, in that case it follows that $p_S \leq p_t(K, D) \leq p_M$, so that the production decisions of both parties are independent of p_t and they equal the centralized decisions: $x^{\mathrm{sub}\ p_t}(K, D) = x^{cen}(K, D)$. The particular choice of $p_t(K, D)$ then guarantees that firm price-only revenue functions equal those under the bargaining model and hence their investment vectors are identical. If the manufacturer's bargaining power is high or the margin difference is small, the existence of an equivalent state-dependent price-only contract is not guaranteed.

4.3 More Complex Contracts

The price-only contracts studied here are the simplest contracts possible. Clearly, one can include more variables into the contract specification. Cachon and Lariviere [6] give an excellent overview of more sophisticated contracts used in the literature and their costs and benefits. These more complex contracts typically specify not only a transfer price p_t, but also some conditions on the transfer quantity x_t, or on the manufacturer's liability of the supplier's excess capacity. Cachon and Lariviere show that these more advanced contracts can, but do not necessarily, improve system coordination and highlight the role of the information structure and the verifiability (and thus enforcement) of the players' actions. In the presence of information asymmetries, complex contracts provide for a powerful signaling device that can improve performance. Tsay [21] has shown that some price-quantity contracts also improve system coordination. While we analyzed only simple contracts, we believe that many of the characteristics of more complex outsourcing contracts will carry over to our subcontracting model.

5 Discussion and Extensions

We have analyzed three contract types to study some important aspects of the subcontracting decision. Our interest was in the financial benefits that subcontracting with these various contract types may offer in an economic environment where market demands are uncertain. Because our main results are already summarized in the abstract and introduction, let us discuss briefly some issues and extensions. It is clear that our analysis is only a first attempt to study the complex practice of subcontracting and outsourcing. Relatively straightforward extensions are the inclusion of specific transaction costs and merging costs. We have assumed that the initiation and management of the subcontracting relationship was costless. A positive cost is directly incorporated so that both parties would enter into the relationship only if the ex-post surplus exceeds the transaction cost. Similarly, one can include merging costs which would explain why both parties not always choose to merge into a single, centralized organization. Another variation is to make both firms more equal 'partners' by dropping the non-negativity constraint on x_t to allow for bi-directional transfers. (This also yields a two-location inventory model with transfers between profit centers.)

In addition to analyzing more complex price-quantity contracts and information structures as discussed in Section 4.3, other involved extensions to the model would be to allow for demand-dependent sales prices (and thus margins) by incorporating downward sloping demand curves (our firms are assumed to be price takers). Such an approach yields a duopoly model more in-line with traditional economic theory and allows us to incorporate tactical pricing decisions. This generalization, however, comes at a considerable cost. One not only looses the connection to the traditional newsvendor model and its intuitive, graphical interpretation, but the competitive pricing decision under uncertainty greatly increases the complexity of the analysis[2]. Allowing for non-exclusive market access is an easier extension that, we believe, will not change the qualitative insights obtained here. Finally, the time-horizon can be extended to a multi-period setting to study the effect of predictable temporal demand variations, such as over a product life cycle (stochastic temporal variations most likely will lead to a production smoothing effect as studied by Kamien and Li [14]).

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[2] Allowing for inter-firm subcontracting transfers would amount to putting yet another layer of complexity on the competitive investment-pricing model that we studied in [24].

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6 Appendix

All first order optimality equations (OE) are of the form $\mathrm{E}\lambda = \Lambda\overline{P} = c$,where the 2×6 matrix Λ is function only of p_t, p_M and p_S, while the vector $\overline{P}$ is function only of K (and of parameters in the probability distribution). The structure of the OEs (or capacity reaction curves) and uniqueness of an optimal solution will be established using partial derivatives which are found by implicitly differentiating one or both OE. Let x represent a cost or margin parameter of interest. Total differentiation of the OE yields:

$$\frac{d}{dx}\mathrm{E}\lambda = \frac{d}{dx}\Lambda\overline{P} = \left(\frac{\partial}{\partial x}\Lambda\right)\overline{P} + J\frac{\partial}{\partial x}K,$$

where J is the Jacobian matrix of the OE: $J_{ij} = \frac{\partial \mathrm{E}\lambda_i}{\partial K_j} = \frac{\partial V_i}{\partial K_j \partial K_i}$, which can be calculated explicitly:

$$J = \Lambda \left[\begin{array}{cc} \frac{\partial}{\partial K_M}\overline{P} & \frac{\partial}{\partial K_S}\overline{P} \end{array}\right] = \Lambda\left(\nabla_K \overline{P}'\right)',$$

where the 2×6 matrix $\nabla_K \overline{P}'$ can be expressed in terms of the line integrals L_{ij} of the probability density $f(\cdot)$ over the boundary between domains Ω_i and Ω_j and $L_{ij,kl} = L_{ij} + L_{kl}$:

$$\nabla_K \overline{P}' = \left[\begin{array}{cccccc} L_{16} - L_{01} & L_{23} & L_{34} - L_{23} & -L_{34} & -L_{56} & L_{56} - L_{16} \\ L_{16} & -L_{02} & L_{34} - L_{36} & -L_{34} - L_{45} & L_{45} - L_{56} & L_{56} + L_{36} - L_{16} \end{array}\right].$$

For example: $L_{23} = \int_{K_S}^{\infty} f(K_M, D_S)dD_S$. Thus, all effort is reduced to showing that $J = \Lambda\left(\nabla_K \overline{P}'\right)'$ is invertible which then yields

$$\frac{\partial}{\partial x}K = J^{-1}\frac{dc}{dx} - J^{-1}\left(\frac{\partial}{\partial x}\Lambda\right)\overline{P}. \tag{8}$$

Thus, letting $x = c_i$ we directly have that

$$\left[\begin{array}{cc} \frac{\partial}{\partial c_M}K & \frac{\partial}{\partial c_S}K \end{array}\right] = J^{-1}, \tag{9}$$

and the slope of $k_i(\cdot)$, the OE for K_i given K_j, follows from totally differentiating the i'th OE: $\frac{\partial}{\partial K_i}\mathrm{E}\lambda_i \frac{dk_i}{dK_j} + \frac{\partial}{\partial K_j}\mathrm{E}\lambda_i = 0$ or

$$\frac{dk_i}{dK_j} = -\frac{\frac{\partial^2}{\partial K_j \partial K_i}V_i}{\frac{\partial^2}{\partial K_i^2}V_i} = -\frac{J_{ij}}{J_{ii}}.$$

6.1 Centralized Reference Scenario

The optimal solution K^{cen} is at the intersection of the two OE curves. We have that

$$J^{cen} = \begin{cases} \left[\begin{array}{cc} -(p_M - p_S)L_{34,56} - p_S L_{16,23} & -(p_M - p_S)L_{34,56} - p_S L_{16} \\ -(p_M - p_S)L_{34,56} - p_S L_{16} & -(p_M - p_S)L_{34,56} - p_S L_{02,16} \end{array}\right] & \text{if } p_S \leq p_M, \\ \left[\begin{array}{cc} -p_M L_{16,23} & -p_M L_{16} \\ -p_M L_{16} & -p_S L_{02} - (p_S - p_M)L_{45,36} - p_M L_{16} \end{array}\right] & \text{if } p_M < p_S. \end{cases}$$

All entries in J are nonpositive with $J_{11} \leq J_{12} \leq 0$ and $J_{22} \leq J_{21} \leq 0$ so that $|J| \geq 0$ and

$$\begin{aligned} |J| &= \begin{cases} (p_M - p_S)p_S L_{34,56}L_{02,23} + p_S^2(L_{16}L_{02} + L_{23}L_{02} + L_{23}L_{16}) & \text{if } p_S \leq p_M, \\ p_M p_S L_{16,23}L_{02} + p_M(p_S - p_M)L_{16,23}L_{45,36} + p_M^2 L_{23}L_{16} & \text{if } p_M < p_S. \end{cases} \\ -1 &\leq \frac{dk_i}{dK_j} = -\frac{J_{ij}}{J_{ii}} \leq 0. \end{aligned}$$

Clearly, if $c_M > c_S$, it is optimal to invest only in S-capacity: $\nu_M > 0$ so that $k_M^{cent}(\cdot) = 0$ and $k_S^{cent}(\cdot) = K_S^{cent}$. If $p_S \leq p_M$, $p_S P_{236} + p_M P_{45} = c_S$. Because $P_{02} = 0$, either L_{16} and/or $L_{34,56}$ are positive so that OE_S is strict concave at K_S^{cen} ($J_{22} < 0$), ergo uniqueness. If $p_M < p_S$, $p_S P_{234} + p_M P_{56} = c_S$ and either $L_{34,36}$ and/or L_{16} are positive, again showing uniqueness.

Otherwise, if $c_M < c_S$, we invest in both capacities and at least one of the terms in $|J|$ is positive so that V^{cen} is strict concave at the unique optimal K^{cen}. We can compute some points of the centralized curves:

- If $K_S = 0$ and $c_M < c_S$, then $P_{01356} = 0, L_{23} = L_{34}, L_{16,56} = 0$ and $p_M P_4 = p_M P(D_M > K_M^{cent}) = c_M$. Thus,

$$k_M^{cent}(0) = K_M^{solo} \text{ and } \frac{dk_M}{dK_S} = \begin{cases} -\frac{p_M - p_S}{p_M} > -1 & \text{if } p_S \le p_M, \\ 0 & \text{if } p_S > p_M. \end{cases}$$

- If $K_S \to \infty$ and $c_M < c_S$, then $P_{23456} = 0$ so that $L_{34,56,16} = 0$ and $\mathrm{E}\lambda_M = 0 < c_M$ so that

$$k_M(\infty) = 0 \text{ and } \frac{dk_M(\infty)}{dK_S} = 0,$$

a situation that remains if K_S decreases as long as $P_{01}((0, K_S)) = 1$. Clearly, this minimal K_S increases in correlation and variability.

- If $K_M = 0$, then $P_{02} = 0$ and $L_{02} = 0$. Thus,

$$\begin{cases} \frac{dk_S^{cent}(0)}{dK_M} = -1 & \text{if } p_S \le p_M, \\ -1 \le \frac{dk_S^{cent}(0)}{dK_M} \le 0 & \text{if } p_S > p_M. \end{cases}$$

- If $K_M \to \infty$, then $P_{13456} = 0$ so that $L_{34,56,16} = 0$ and $p_S P_2 = c_S$. Thus,

$$k_S^{cent}(\infty) = G_S(\frac{c_S}{p_S}) = K_S^{solo} \text{ and } \frac{dk_S^{cent}(\infty)}{dK_M} = 0,$$

a situation that remains if K_M decreases as long as $P_{456}((K_M, k_S(\infty)) = 0$. Clearly, this minimal K_M increases in variability.

6.2 Subcontracting with Price-Only Contracts

The Jacobian becomes

$$J = \begin{cases} \begin{bmatrix} -(p_M - p_t)L_{34,56} - p_t L_{01,23} & -(p_M - p_t)L_{34,56} \\ -(p_t - p_S)L_{34,56} - p_S L_{16} & -(p_t - p_S)L_{34,56} - p_S L_{02,16} \end{bmatrix} & \text{if } p_S \le p_t \le p_M, \\ \begin{bmatrix} -p_t L_{01} - p_M L_{23} - (p_M - p_t)L_{16} & -(p_M - p_t)L_{16} \\ -p_t L_{16} & -p_S L_{02} - (p_S - p_t)L_{45,36} - p_t L_{16} \end{bmatrix} & \text{if } p_t < \min(p). \end{cases}$$

6.2.1 Uniqueness of the solution K^{sub}

All entries in J are nonpositive with $J_{11} \le J_{12} \le 0$ and $J_{22} \le J_{21} \le 0$ so that $|J| \ge 0$ and

$$|J| = \begin{cases} (p_M - p_t)p_S L_{34,56}L_{02} + (p_t - p_S)p_t L_{01,23}L_{34,56} + p_t p_S L_{01,23}L_{02,16} & \text{if } p_S \le p_t \le p_M, \\ (p_M - p_t)p_S L_{02}L_{16} + (p_M - p_t)(p_S - p_t)L_{16}L_{45,36} + p_t(p_S - p_t)L_{01}L_{45,36} & \\ +p_M(p_S - p_t)L_{23}L_{45,36} + p_t p_S L_{01}L_{02} + p_t^2 L_{01}L_{16} + p_M p_S L_{23}L_{02} + p_M p_t L_{23}L_{16} & \text{if } p_t < \min(p). \end{cases}$$

$$-1 \le \frac{dk_i}{dK_j} = -\frac{J_{ij}}{J_{ii}} \le 0.$$

Existence of an intersection follows from the relative position of axis crossings and asymptotes:

- If $K_S = 0$, then $P_{01356} = 0, L_{23} = L_{34}, L_{01,16,56} = 0$ and $p_M P_4 = p_M P(D_M > K_M) = c_M$. Thus,

$$k_M(0) = G_M(\frac{c_M}{p_M}) = K_M^{solo} \text{ and } \frac{dk_M(0)}{dK_S} = \begin{cases} -\frac{p_M - p_t}{p_M} > -1 & \text{if } p_S \le p_t \le p_M, \\ 0 & \text{if } p_t < \min(p). \end{cases}$$

($\frac{dk_M(0)}{dK_S}$ remains 0 as K_S increases with low p_t until P_1 becomes positive. Clearly, this maximal K_S decreases in correlation and variability.)

- If $K_S \to \infty$, then $P_{23456} = 0$ so that $L_{34,56,16} = 0$ and $\mathrm{E}\lambda_M = p_t P_1 \le p_t$. Thus, if $p_t < c_M$, we have $k_M(\infty) = 0$, else
$$k_M(\infty) = G_M(\frac{c_M}{p_t}) < K_M^{solo} \text{ and } \frac{dk_M(\infty)}{dK_S} = 0,$$
a situation that remains if K_S decreases as long as $P_{01}\left((k_M(\infty), K_S)\right) = 1$. Clearly, this minimal K_S increases in correlation and variability. Note that $k_M(\cdot)$ is continuous in p_t for $p_S < p_t < p_M$, except at $p_t = c_M$ if D_M is bounded from below by a positive number with probability one (the demand density is zero at $D_M = 0$).

- If $K_M = 0$, then $P_{02} = 0$ so that $L_{02} = 0$. For high p_t we have that $p_S P_{36} + p_t P_{45} = c_S$ and because $p_S < p_t$, we have that
$$\frac{dk_S(0)}{dK_M} = -1.$$
With small variability, we have that $k_S(0) \simeq D_+$ (exact: $P(D_+ > k_S(0)) = \frac{c_S}{p_S}$. Indeed, if $K_S \ll (\gg) D_+$, we would have that $P_{3456} = 1(0)$, which cannot satisfy OE_S.) For low p_t we have that $p_S P_{34} + p_t P_{56} = c_S$. If $p_t < c_S$, then $P_{34} > 0$. If D has low variability in the sense that $P(\Omega_1(K = (0, K_S^{solo}))) = 0$, then $k_2(\cdot)$ is discontinuous at $p_t = c_2$ and we have that
$$\begin{aligned} k_2(0) &\approx D_+ \text{ (exactly: } P_{56} = \frac{c_S}{p_t}) \text{ and } \frac{dk_S(0)}{dK_M} = -1 \text{ IF } p_t > c_S, \\ k_S(\cdot) &= K_S^{solo} \approx D_S \text{ and thus } \frac{dk_S(0)}{dK_M} = 0 \text{ IF } p_t < c_S. \end{aligned}$$
If D has high variability, $0 \le \frac{dk_S(0)}{dK_M} \le -1$.

- If $K_M \to \infty$, then $P_{13456} = 0$ so that $L_{34,56,16} = 0$ and $p_S P_2 = c_S$. Thus,
$$k_S(\infty) = G_M(\frac{c_S}{p_S}) = K_S^{solo} \text{ and } \frac{dk_S(\infty)}{dK_M} = 0,$$
a situation that remains if K_M decreases as long as $P_{456}\left((K_M, k_S(\infty)\right) = 0$. Clearly, this minimal K_M increases in variability.

Uniqueness of K^{sub} follows from $-1 < \frac{dk_M}{dK_S}$ at intersection (assume high p_t, low p_t is similar)

- If $p_M \ge p_t > c_M : 0 < P_{13645} < 1$ and because P is a continuous measure we have that $L_{23,01} > 0$ so that V_M is strict concave at the optimal K_M and thus the reaction curve $k_M(\cdot)$ is unique. Moreover
$$-1 < \frac{dk_M}{dK_S} \le 0 \text{ (and } \frac{dk_M}{dK_S} = 0 \text{ if } P_{45} = 0).$$

- If $p_M > c_M \ge p_t : 0 < P_{45} < 1$ so that $L_{34,56} > 0$. Again the reaction curve $k_M(\cdot)$ is unique but now, as long as $k_M > 0$:
$$-1 \le \frac{dk_M}{dK_S} < 0 \text{ (and } \frac{dk_M}{dK_S} = -1 \text{ if } P_{012} = 0).$$
At the intersection K^{sub} we have that $-1 < \frac{dk_M}{dK_S}$ which shows uniqueness (indeed $P_{012} = 0$ would imply $P_{3456} = 1$, which cannot be a solution to $\mathrm{OE}_S : p_S P_{36} + p_t P_{45} \ge \min(p_S, p_t) = p_S > c_S$).

Similarly for firm 2's reaction curves ($p_S P_{236} + p_t P_{45} = c_S$), if follows that

- Because $p_t > p_S > c_S : 0 < P_{236}, P_{45} < 1$ and thus $0 < P_{01} < 1$ and $L_{02,16} > 0$ so that V_S is strict concave at the optimal K_S and thus the reaction curve $k_S(\cdot)$ is unique. Moreover
$$-1 \le \frac{dk_S}{dK_M} \le 0 \text{ (and } \frac{dk_S}{dK_M} = -1 \text{ if } P_2 = 0 \text{ and } \frac{dk_S}{dK_M} = 0 \text{ if } P_{13456} = 0).$$

Given that the two reaction curves are unique with $-1 \le \frac{dk_i}{dK_j} \le 0$, and the relative axis crossings are as given higher together with $-1 < \frac{dk_M}{dK_S}$ at any solution to the OE, it follows that hey have a unique intersection which is a stable, and thus Nash, equilibrium.

6.2.2 Sensitivity of K^{sub}

The intersection point K is the unique solution to the OE and it follows from the OE that one will never invest to cover all demand with probability 1. In other words, if $K > 0$, then $0 < P_{01} < 1$ and at least one of the terms in $\det(J)$ is positive so that $|J| > 0$ and J is invertible:

$$J^{-1} = \begin{cases} |J|^{-1}\begin{bmatrix} -(p_t - p_S)L_{34,56} - p_S L_{02,16} & (p_M - p_t)L_{34,56} \\ (p_t - p_S)L_{34,56} + p_S L_{16} & -(p_M - p_t)L_{34,56} - p_t L_{01,23} \end{bmatrix} & \text{if } p_S \le p_t \le p_M, \\ |J|^{-1}\begin{bmatrix} -p_S L_{02} - (p_S - p_t)L_{45,36} - p_t L_{16} & (p_M - p_t)L_{16} \\ p_t L_{16} & -p_t L_{01} - p_M L_{23} - (p_M - p_t)L_{16} \end{bmatrix} & \text{if } p_t < \min(p). \end{cases}$$

$$= -\begin{bmatrix} \alpha_1 + \alpha_3 & -\alpha_2 \\ -\alpha_1 & \alpha_2 + \alpha_4 \end{bmatrix}.$$

Because $\frac{\partial K_i}{\partial c_j} = J_{ij}$, we have that both capacities are imperfect substitutes w.r.t. the marginal cost vector. Partials w.r.t. the margins are

$$\frac{\partial}{\partial p_M}K = \begin{cases} -J^{-1}\begin{bmatrix} 0&0&0&1&1&0 \\ 0&0&0&0&0&0 \end{bmatrix}P = \begin{bmatrix} \alpha_1+\alpha_3 \\ -\alpha_1 \end{bmatrix}P_{45} & \text{if } p_S \le p_t \le p_M, \\ -J^{-1}\begin{bmatrix} 0&0&1&1&1&1 \\ 0&0&0&0&0&0 \end{bmatrix}P = \begin{bmatrix} \alpha_1+\alpha_3 \\ -\alpha_1 \end{bmatrix}P_{3456} & \text{if } p_t < \min(p). \end{cases}$$

$$\frac{\partial}{\partial p_S}K = \begin{cases} -J^{-1}\begin{bmatrix} 0&0&0&0&0&0 \\ 0&1&1&0&0&1 \end{bmatrix}P = \begin{bmatrix} -\alpha_2 \\ \alpha_2+\alpha_4 \end{bmatrix}P_{236} & \text{if } p_S \le p_t \le p_M, \\ -J^{-1}\begin{bmatrix} 0&0&0&0&0&0 \\ 0&1&1&1&0&0 \end{bmatrix}P = \begin{bmatrix} -\alpha_2 \\ \alpha_2+\alpha_4 \end{bmatrix}P_{234} & \text{if } p_t < \min(p). \end{cases}$$

$$\frac{\partial}{\partial p_t}K = \begin{cases} -J^{-1}\begin{bmatrix} 1&0&1&0&0&1 \\ 0&0&0&1&1&0 \end{bmatrix}P = \begin{bmatrix} (\alpha_1+\alpha_3)P_{136} - \alpha_2 P_{45} \\ -\alpha_1 P_{136} + (\alpha_2+\alpha_4)P_{45} \end{bmatrix} & \text{if } p_S \le p_t \le p_M, \\ -J^{-1}\begin{bmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&1&1 \end{bmatrix}P = \begin{bmatrix} (\alpha_1+\alpha_3)P_1 - \alpha_2 P_{56} \\ -\alpha_1 P_1 + (\alpha_2+\alpha_4)P_{56} \end{bmatrix} & \text{if } p_t < \min(p). \end{cases}$$

While $\frac{\partial}{\partial p_t}K$ cannot be signed in general, we do have that $\frac{\partial}{\partial p_t}K_+ > 0$.

6.2.3 Sensitivity of V^{sub}

We have that $\frac{dV_i}{dx} = \frac{\partial V_i}{\partial K_M}\frac{\partial K_M}{\partial x} + \frac{\partial V_i}{\partial K_S}\frac{\partial K_S}{\partial x} + \frac{\partial V_i}{\partial x}$, where $\frac{\partial V_i}{\partial K_i} = 0$ under optimal investment. The cross-partial $\frac{\partial V_i}{\partial K_j} = \frac{\partial}{\partial K_j}\mathrm{E}\pi_i = \mathrm{E}\frac{\partial}{\partial K_j}\pi_i = \mathrm{E}\lambda_{i,j}$ can be computed as before by the weighted average of the constant $\lambda_{i,j}^j$ in each domain l:

$$\frac{\partial V_M^{sub}}{\partial K_S} = \mathrm{E}\lambda_{1,2} = (p_M - p_t)P_{45} \ge 0$$

$$\frac{\partial V_S^{sub}}{\partial K_M} = \mathrm{E}\lambda_{2,1} = \begin{cases} -p_t P_1 - (p_t - p_S)P_{36} \le 0 & \text{if } p_S \le p_t \le p_M, \\ -p_t P_1 \le 0 & \text{if } p_t < \min(p). \end{cases}$$

Denoting $\mathrm{E}\lambda_{1,2} = \beta_5 \ge 0$ and $\mathrm{E}\lambda_{2,1} = -\beta_6 \le 0$, we get

$$\frac{\partial V_M^{sub}}{\partial c_M} = \alpha_1\beta_5 - K_1^{sub} = \beta_1 - K_M^{sub}, \quad \frac{\partial V_S^{sub}}{\partial c_M} = \beta_6(\alpha_1 + \alpha_3) = \beta_3 \ge 0,$$
$$\frac{\partial V_M^{sub}}{\partial c_S} = -(\alpha_2 + \alpha_4)\beta_5 = -\beta_2 \le 0, \quad \frac{\partial V_S^{sub}}{\partial c_S} = -\beta_6\alpha_2 - K_S^{sub} = -\beta_4 - K_S^{sub} \le 0.$$

As expected $\frac{\partial V_M}{\partial c_S}, \frac{\partial V_S}{\partial c_S}$ are negative and $\frac{\partial V_S}{\partial c_M}$ is positive, while $\frac{\partial V_M}{\partial c_M}$ cannot be signed in general. For price sensitivity consider high transfer prices (the other case is similar but replace the P_{45} by P_{3456}, P_{236} by P_{234}, P_{136} by P_1, and P_{45} by P_{56}):

$$\frac{\partial V_M^{sub}}{\partial p_M} = -\beta_5\alpha_1 P_{45} + \mathrm{E}x_{1+t} = -\beta_1 P_{45} + \mathrm{E}x_{1+t}, \quad \frac{\partial V_S^{sub}}{\partial p_M} = -\beta_6(\alpha_1 + \alpha_3)P_{45} = -\beta_3 P_{45} \le 0,$$
$$\frac{\partial V_M^{sub}}{\partial p_S} = \beta_5(\alpha_2 + \alpha_4)P_{236} = \beta_2 P_{236} \ge 0, \quad \frac{\partial V_S^{sub}}{\partial p_S} = \beta_6\alpha_2 P_{236} + \mathrm{E}x_S = \beta_4 P_{236} + \mathrm{E}x_S \ge 0.$$
$$\frac{\partial V_M^{sub}}{\partial p_t} = \beta_5\frac{\partial K_S^{sub}}{\partial p_t} - c_M\frac{\partial K_M^{sub}}{\partial p_t} - \mathrm{E}x_t, \quad \frac{\partial V_S^{sub}}{\partial p_t} = -\beta_6\frac{\partial K_M^{sub}}{\partial p_t} - c_S\frac{\partial K_S^{sub}}{\partial p_t} + \mathrm{E}x_t.$$

6.3 Outsourcing Conditions

6.3.1 Low transfer price: $p_t < \min(p)$

Because $K_M = 0$, we have that $P_{02} = 0$ and $P_{13456} = 1$ and the optimality equations yield

$$\begin{aligned} p_t P_1 + p_M P_{3456} &= \bar{c}_M, \\ p_S P_{34} + p_t P_{56} &= c_S, \end{aligned}$$

so that $\frac{c_S}{p_S} \leq P_{3456} \leq \frac{c_S}{p_t}$

$$p_t + \frac{p_M - p_t}{p_S} c_S \leq \bar{c}_M = p_t + (p_M - p_t) P_{3456} \leq p_t + \frac{p_M - p_t}{p_t} c_S.$$

Also,

$$\bar{c}_M = c_S + p_t P_1 + (p_M - p_S) P_{34} + (p_M - p_t) P_{56} \geq^{\text{if } p_M \geq p_S} c_S.$$

Notice that with low levels of uncertainty, one either has

$$\begin{aligned} p_t &< c_S : \bar{K}_S = k_S(0) \simeq D_S \text{ (exactly: } p_S P_{34} = c_S), P_{3456} = 1 \Rightarrow \bar{c}_M = p_M. \\ c_S &< p_t : \bar{K}_S = k_S(0) \simeq D_+ \text{ (exactly: } p_t P_6 = c_S), P_{16} = 1, P_{02345} = 0 \Rightarrow \bar{c}_M = p_t + \frac{p_M}{p_t} c_S - c_S. \end{aligned}$$

As uncertainty increases, $\bar{c}_M$ will *decrease*. Indeed, if $p_t < c_S$, increasing uncertainty will decrease P_{3456} from 1 and increase P_1, but P_1 has lower coefficient $p_t < p_M$ in the definition of $\bar{c}_M$. If $p_t > c_S$, increasing uncertainty will decrease P_6 and increase P_{534}. From OE 2 we see that P_6 will decrease more than P_{345} will increase ($p_S > p_t$); thus P_1 will also increase, but again less than the decrease in P_6, so that $\bar{c}_M$ will decrease because $p_t < p_M$.

6.3.2 High transfer price: $p_S < p_t < p_M$

Because $K_M = 0$, we have that $P_{02} = 0$ and $P_{13456} = 1$ and the optimality equations yield

$$\begin{aligned} p_t P_{136} + p_M P_{45} &= \bar{c}_M, \\ p_S P_{36} + p_t P_{45} &= c_S, \end{aligned}$$

so that $0 \leq P_{45} \leq \frac{c_S}{p_t}$

$$p_t \leq \bar{c}_M = p_t + (p_M - p_t) P_{45} \leq p_t + \frac{p_M - p_t}{p_t} c_S.$$

Again, with limited levels of uncertainty, one can only have ($p_t > p_S > c_S$):

$$\bar{K}_S = k_S(0) \simeq D_+ \text{ (exact: } p_S P_6 = c_S), P_{16} = 1, P_{02345} = 0 \Rightarrow \bar{c}_M = p_t.$$

As uncertainty increases, P_{16} will decrease from 1 and P_{345} will grow, leading to an *increase* in $\bar{c}_M$ because $p_t < p_M$. Finally, notice that $\bar{c}_M$ is discontinuous at $p_t = p_S$.

6.3.3 Incomplete Contracts (Bargaining)

Because $K_M = 0$, we have that $P_{02} = 0$ and $P_{13456} = 1$ and the optimality equations yield (assuming $p_S < p_M$):

$$\begin{aligned} \bar{\theta} p_M P_1 + (\bar{\theta} p_M + \theta p_S) P_{36} + p_M P_{45} &= \bar{c}_M, \\ p_S P_3 + (\bar{\theta} p_M + \theta p_S) P_4 + \bar{\theta} p_M P_5 + \bar{\theta} p_S P_6 &= c_S, \end{aligned}$$

so that

$$\bar{\theta} p_M + \theta c_S \leq \bar{c}_M = \bar{\theta} p_M + \theta c_S + \theta^2 \left((p_M - p_S) P_4 + p_M P_5 + p_S P_6 \right) \leq \bar{\theta} p_M + \theta c_S + \frac{\theta^2}{\bar{\theta}} c_S = \bar{\theta} p_M + \frac{\theta}{\bar{\theta}} c_S.$$

If $p_S \geq p_M$, we have that

$$\bar{c}_M = p_M P_1 + (p_M + \theta p_M) P_{3456} \geq p_M.$$