Economic Theory **12**, 293–312 (1998)

Economic Theory

Recursive utility and preferences for information★

Costis Skiadas

J.L. Kellogg Graduate School of Management, Northwestern University,
Evanston, IL 60208-2001, USA (e-mail: c-skiadas@nwu.edu)

Received: February 18, 1997; revised version: July 18, 1997

Summary. This paper presents an axiomatic foundation for recursive utility that captures the role of the timing of resolution of uncertainty without relying on exogenously specified objective beliefs. Two main representation results are proved. In the first one, future utility enters the recursion through the type of general aggregators considered in Skiadas (1997a), and as a result the formulation is purely ordinal and free of any probabilities. In the second representation these aggregators are conditional expectations relative to subjective beliefs. A new recursive representation incorporating disappointment aversion is also suggested. The main methodological innovation of the paper derives from the fact that the basic objects of choice are taken to be pairs of state-contingent consumption plans and information filtrations, rather than the temporal (objective) lotteries of the existing literature. It is shown that this approach has the additional benefit of being directly applicable to the continuous-time version of recursive utility developed by Duffie and Epstein (1992).

JEL Classification Numbers: D81, D91.

1 Introduction

The main purpose of this paper is to provide a simple axiomatic foundation for recursive utility that captures the role of the timing of resolution of uncertainty, without relying on exogenously specified objective beliefs. We introduce a setting in which information is modeled through filtrations, rather than temporal lotteries, and preferences for the timing of resolution are expressed as the monotonicity of a utility function with respect to its

★ Earlier versions of this paper were titled "Time-coherent choice and preferences for information," or were part of old drafts of the paper "Conditioning and aggregation of preferences." The paper has benefited from discussions with Soo Hong Chew, Darrell Duffie, Larry Epstein, Tan Wang, as well as the feedback of an anonymous referee.

filtration argument. This methodology has the advantage that it avoids elaborate choice spaces of nested objective probability distributions, dealing directly with state-contingent consumption plans (as in modern general equilibrium theory) and allowing for subjective probabilities revealed through choice. Another advantage of the approach is that it extends readily to continuous time formulations. This fact will be demonstrated by extending the Kreps and Porteus (1978) analysis of preferences for the timing of resolution of uncertainty to the continuous time setting of Duffie and Epstein (1992).

The importance and structure of preferences that are not necessarily temporally additive has been studied extensively in a literature surveyed by Epstein (1992), and Karni and Schmeidler (1991). It is, for example, well known that utility additivity with respect to time and states of nature is overly restrictive in expressing reasonable notions of risk aversion in temporal settings. A familiar illustration, a version of which is reported by Duffie and Epstein (1992), involves the ranking of two temporal lotteries. The payoffs of lottery A are determined by tossing a coin every day for ten years, and paying out \$100 or nothing immediately after each toss, depending on whether the outcome is heads or tails. Lottery B also pays out \$100 or nothing each day over ten years, but there is only one coin toss: if the outcome is heads all payments are \$100, and if the outcome is tails all payments are zero. In terms of direct preferences, most people would agree that lottery A is less "risky" than lottery B. Yet, any utility function that is additive with respect to time and states of nature would assign the same utility level to both alternatives. Recursive utility can be viewed as a way of overcoming this type of difficulty, while retaining dynamic consistency. As Epstein (1992) explains in his survey, under additive temporal preferences notions of intertemporal substitution and risk aversion are inflexibly linked to each other, while recursive utility allows an added degree of flexibility through the curvature of an intertemporal aggregator.

Another consideration that arose in the literature of dynamic choice, following "induced-utility" arguments by Mossin (1969), Dreze and Modigliani (1972), and Spence and Zeckhauser (1972), is that it is not sufficient to consider preferences over state-contingent consumption over time alone, the manner in which uncertainty resolves over time must also be a factor that determines utility. For example, continuing the illustration of Duffie and Epstein (1992), consider a third lottery C that is the same as A, except that all coins are tossed today (without changing the timing of payments). To distinguish between lotteries A and C, more than the specification of state-contingent payments is needed. Kreps and Porteus (1978) were the first to consider a direct axiomatic derivation of recursive utility that was sufficiently structured to model the role of the timing of resolution of uncertainty. Kreps and Porteus showed that the concavity (convexity) of an intertemporal aggregator in their setting induces preferences for late (early) resolution of uncertainty. Consequently, notions of risk-aversion in temporal settings naturally lead to preferences for the timing of resolution. Intuitively, a risk-averse person may prefer a less informed signal of future consequences, fearing the possibility of bad news, a situation all too familiar to anyone who has delayed a visit to the doctor, seemingly against one's best interest. Of course the same risk-averse individual faces benefits of planning from early resolution. In an optimization problem, the latter effect can overcome the direct utility effect. The work of Kreps and Porteus has been extended by Chew and Epstein (1989, 1991), Grant, Kajii, and Polak (1996), and others.

Following Kreps and Porteus, papers on dynamic choice that incorporate the role of the timing of resolution model objects of choice as temporal lotteries, therefore assuming objectively given probabilities, and resulting in technical structures of considerable complexity. Johnsen and Donaldson (1985) on the other hand showed that recursive utility can be simply formulated by placing time-consistency restrictions on preferences over state-contingent consumption plans. Their setting is not rich enough to capture the role of the timing of resolution of uncertainty, but they conjecture that the Kreps-Porteus analysis should have an intuitive counterpart in their setting. Their conjecture is confirmed and extended in this paper. In order to express the fact that different choices can result in different information streams, we extend the Johnsen-Donaldson formulation so that objects of choice are pairs of state-contingent consumption plans and information filtrations. Preferences for early or late resolution can then be thought of simply as the monotonicity of a utility in its filtration argument, a property that can be characterized in a way analogous to that of Kreps and Porteus.

The paper contains two main representation theorems for recursive utility. The first result builds on the notion of aggregation developed in Skiadas (1997a), to derive a recursion of the form $V_t = f_t(c_t, A_t[V_{t+1}])$, where V_t is the time-t continuation utility (or, in the language of Koopmans (1960), prospective utility), c_t represents time-t consumption, and A_t is a conditional aggregator given time-t information that is not necessarily additive. In this way a purely ordinal theory is obtained, without the usual strong structural assumptions required for cardinal or subjective probability representations. Another representation theorem shows that any theory of subjective expected utility can be used as a basis for deriving the above recursive representation, but with A_t being a conditional expectation relative to subjective probabilities. The main part of the paper is on utility representations that imply the irrelevance of unrealized alternatives, and history independence. None of these conditions, however, is inherently a requirement of the paper's approach. The final section presents a brief discussion of possible extensions that allow for aspects of preferences such as disappointment aversion (extending the formulation in Skiadas (1997a,b)) or habit formation.

The remainder of the paper is organized in five sections and two appendices. Section 2 introduces the primitives of a discrete-time model, and the basic notion of "coherence." Section 3 defines recursive utility in such a model, and discusses preferences for the timing of resolution of uncertainty. Section 4 contains two ordinal representation theorems, while Section 5 presents a representation incorporating subjective beliefs. Section 6 discusses extensions of the model. Appendix A is on the continuous-time case, extending the formulation of Duffie and Epstein (1992) to incorporate the role of the timing of resolution of uncertainty. Appendix B contains proofs.

2 Primitives and the notion of coherence

We begin with a discrete-time, finite state space model. The primitives introduced in this section will be taken as given throughout the main part of the paper.

Uncertainty is represented by the finite[1] state space Ω. An *event* is any nonempty subset of Ω. The set of all events is denoted $\mathscr{E}$. Time is represented by the set $\mathscr{T} = \{0, \ldots, T\}$ whose elements we call *times*. We model information through algebras and filtrations. An *algebra* is any set of subsets of Ω that is closed under unions and complementation. Given any algebra $\mathscr{F}$, there is a unique partition of Ω that generates $\mathscr{F}$, denoted $\mathscr{F}^0$. A *filtration* is any sequence of algebras of the form $\{\mathscr{F}_t : t \in \mathscr{T}\}$ such that $\mathscr{F}_t \subseteq \mathscr{F}_{t+1}$ for all $t < T$, and $\mathscr{F}_0 = \{\emptyset, \Omega\}$. Consumption bundles are represented by elements of a set $\mathscr{C}$. A *consumption process* is any function of the form $c : \Omega \times \mathscr{T} \to \mathscr{C}$, and is said to be *adapted* to the filtration $\{\mathscr{F}_t\}$ if, for every time t, the random variable $c_t = c(\cdot, t)$ is $\mathscr{F}_t$-measurable (meaning that for every $F \in \mathscr{F}_t^0$, the restriction of c_t to F is constant).

The decision maker expresses preferences over a set X, whose elements we call *acts*. With every act $x \in X$, we associate two objects: a consumption process $c(x)$, and a filtration $\{\mathscr{F}_t(x) : t \in \mathscr{T}\}$. The random variable $c_t(x)$ represents time-t state-contingent consumption resulting from act x, while the algebra $\mathscr{F}_t(x)$ represents the time-t information available to the decision maker given choice x. We assume throughout that, for every $x \in X$, $c(x)$ is *adapted* to $\{\mathscr{F}_t(x)\}$. Conversely, given any filtration $\{\mathscr{F}_t\}$, and any consumption process c that is adapted to $\{\mathscr{F}_t\}$, there exists an act $x \in X$ such that $c(x) = c$ and $\mathscr{F}_t(x) = \mathscr{F}_t$ for all $t \in \mathscr{T}$. Our later restrictions on preferences will imply that an act x can be identified with the pair of the consumption process and filtration it generates.

Given any event F, we define $X_t^F = \{x \in X : F \in \mathscr{F}_t(x)\}$, the set of all acts under which the truth of F or its complement is known at time t. We take as primitive the set $\{\succeq_t^F : F \in \mathscr{E}, t \in \mathscr{T}\}$, where $\succeq_t^F$ is a preference order (that is, a complete and transitive relation) on X_t^F, for every $F \in \mathscr{E}$ and $t \in \mathscr{T}$. We denote by $\succ_t^F$ and $\sim_t^F$ the asymmetric and symmetric parts of $\succeq_t^F$, respectively. The interpretation of the statement $x \succeq_t^F y$ is that the decision maker regards the overall consequences of x realized on event F and during the time interval $\{t, \ldots, T\}$ at least as desirable as the corresponding consequences of y. We refer to $\succeq_t^F$ as the time-t *conditional preference* given F. Of course, the only non-trivial time-zero conditional preference is $\succeq_0^\Omega$.

[1] The finiteness assumption is made for simplicity of exposition. The analysis extends easily to the infinite case using the methods of Appendix A in Skiadas (1997a,b).

Throughout the paper, conditional preferences will be assumed to satisfy the following basic monotonicity condition, for all $t \in \{1, \ldots, T\}$:

A0. (Event Coherence) For any disjoint $F, G \in \mathscr{E}$, and $x, y \in X_t^F \cap X_t^G$,
(a) $x \succeq_t^F y$ and $x \succeq_t^G y$ implies $x \succeq_t^{F \cup G} y$, and
(b) $x \succ_t^F y$ and $x \succ_t^G y$ implies $x \succ_t^{F \cup G} y$.

Event coherence expresses the fact that if the consequences of x over a given time interval are preferred to the corresponding consequences of y in each of two mutually exclusive scenarios, F and G, then the same is true in the joint scenario $F \cup G$. With regard to part (a) of A0 the word "preferred" should be interpreted in the weak sense (preferred or indifferent), and with regard to (b) it should be interpreted in the strong sense. (Notice that we do not assume the stronger statement: $x \succ_t^F y$ and $x \succeq_t^G y$ implies $x \succ_t^{F \cup G} y$, which would correspond to what is called "strict coherence" in Skiadas (1997a,b).) The interpretation and significance of event coherence is further discussed in Skiadas (1997a,b), where the simpler term "coherence" is used instead, since only atemporal choice is considered there.

Another important principle throughout the paper is what we call time coherence, a notion analogous to event coherence, but with time intervals playing the role of events. To define time coherence, suppose that I_1 and I_2 are two consecutive time intervals, and let I be their union. Given any two acts, x, y, and any event F, suppose that, on the event F, the consequences of x during I_i are preferred to the consequences of y during I_i, for both values of i. Time coherence stipulates that it must then be the case that, on event F, the overall consequences of x during I are preferred to the overall consequences of y during I. Moreover, if, on event F, the preferences of x on I_i are strongly preferred to those of y on I_i for some i, then, on event F, the consequences of x on I are strongly preferred to the consequences of y on I. (Time coherence is therefore analogous to the formal definition of strict coherence in Skiadas (1997a,b), with time intervals playing the role of events.)

While time coherence is easily formalized by defining preferences conditioned on time intervals (as well as events), for simplicity, in this paper we use the concept informally as a guide in formulating our postulates. As in Skiadas (1997a,b), strong implications, such as separability or recursivity, are obtained if coherence is coupled with an interpretation of the word "consequence," a concept that will also be used at an informal level. For the main results of the paper, a consequence at a given state and time will be assumed to depend only on present and future consumption and information, implying both history independence and the irrelevance of unrealized alternatives. In the final section we outline extensions in which consequences can include subjective states of mind, such as feelings of disappointment or habits, resulting in recursive representations that, while satisfying event and time coherence, they need no longer satisfy the irrelevance of unrealized alternatives or history independence.

3 Recursive utility

A *utility representation* of the conditional preference family $\{\succeq_t^F\}$ is any set of the form $\{U_t^F : F \in \mathscr{E}, t \in \mathscr{T}\}$, where $U_t^F : X_t^F \to \mathbb{R}$ is a utility representation of $\succeq_t^F$ (that is, $x \succeq_t^F y \Leftrightarrow U_t^F(x) \geq U_t^F(y)$, for all $x, y \in X_t^F$). Given any utility representation $\{U_t^F\}$, and any algebra $\mathscr{F}$, we will use the notation $U_t^{\mathscr{F}}$ to denote the real-valued function on $\Omega \times \{x \in X : \mathscr{F} \subseteq \mathscr{F}_t(x)\}$ defined by $U_t^{\mathscr{F}}(\omega, x) = U_t^F(x)$ whenever $\omega \in F \in \mathscr{F}^0$. The random variable $U_t^{\mathscr{F}}(\cdot, x)$ will be denoted $U_t^{\mathscr{F}}(x)$.

As in Skiadas (1997a), conditional utilities will be related through aggregator operators, defined as follows. Let L be the space of all real-valued functions on Ω, and define a *conditional aggregator* given event F to be any function of the form $A[\cdot \mid F] : L \to \mathbb{R}$ with the property that, for any $U, V \in L$, $U = V$ on F implies $A[U \mid F] = A[V \mid F]$. (Throughout the paper, $U = (\text{resp.}, \geq, >) V$ *on* F means $U(\omega) = (\text{resp.}, \geq, >) V(\omega)$ for all $\omega \in \Omega$.) Given a conditional aggregator family $\{A[\cdot \mid F] : F \in \mathscr{E}\}$, and any algebra $\mathscr{F}$, we will denote by $A[\cdot \mid \mathscr{F}]$ the function on L that takes values also in L, and is defined by $A[V \mid \mathscr{F}](\omega) = A[V \mid F]$ whenever $\omega \in F \in \mathscr{F}^0$. A natural candidate for a conditional aggregator is a conditional expectation operator. The general definition allows other possibilities, however, such as a minimum operator corresponding to worst-case analysis of scenarios.

With the language of conditional utilities and aggregators in place, we can now formulate the type of recursive utility representation that is the main object of study of this paper:

Definition 1. *A utility representation* $\{U_t^F\}$ *of* $\{\succeq_t^F\}$ *is said to be recursive if there exist conditional aggregators* $A[\cdot \mid F]$, $F \in \mathscr{E}$, *and functions* $f_t : \mathscr{C} \times I_t \to I_t$, $t \in \mathscr{T}$, where $I_t \subseteq \mathbb{R}$, such that:

(*a*) *For any algebras* $\mathscr{G}$ *and* $\mathscr{F}$, *and any act* x, *we have*

$$\mathscr{G} \subseteq \mathscr{F} \subseteq \mathscr{F}_t(x) \Rightarrow U_t^{\mathscr{G}}(x) = A\left[U_t^{\mathscr{F}}(x) \mid \mathscr{G}\right], \qquad t \in \{1, \ldots, T\} . \quad (1)$$

(*b*) *The functions* f_t *are strictly increasing in their last argument, and the function* $V : \Omega \times \{0, \ldots, T+1\} \times X \to \mathbb{R}$ defined by $V_t(x) = U_t^{\mathscr{F}_t(x)}(x)$ for $t \in \mathscr{T}$ and $V_{T+1}(x) = 0$ satisfies

$$V_t(x) = f_t(c_t(x), A[V_{t+1}(x) \mid \mathscr{F}_t(x)]), \quad t \in \mathscr{T}, \quad x \in X. \quad (2)$$

In the above representation the conditional aggregators $A[\cdot \mid F]$, $F \in \mathscr{E}$, together with the functions f_t, $t \in \mathscr{T}$, which we call *intertemporal aggregators*, completely specify all conditional preferences through (1) and (2). In particular, conditional and intertemporal aggregators specify the decision maker's attitudes towards information.

To discuss attitudes towards information, we define the partial order $\geq$ on the space of acts, by letting, for every $x, y \in X$, $x \geq y$ if $c(x) = c(y)$ and $\mathscr{F}_t(x) \supseteq \mathscr{F}_t(y)$ for all $t \in \mathscr{T}$. The interpretation of the condition $x \geq y$ is that at time zero the decision maker can anticipate no less information in every future date under act x than under act y, while both acts result in identical consumption plans. Uncertainty about future consumption is therefore resolved at least as fast under act x as under act y. The preference order $\succeq_0^\Omega$ is *information seeking* (respectively, *information averse*), if for all $x, y \in X$, $x \geq y$ implies $x \succeq_0^\Omega y$ (respectively, $y \succeq_0^\Omega x$). A decision maker with information seeking preferences (weakly) prefers early rather than late resolution of uncertainty.

To state a counterpart of the Kreps and Porteus (1978) proposition on preferences for the timing of resolution, we first define a notion of convexity relative to general aggregators. Given any conditional aggregator family $\{A[\cdot \mid F]\}$, we say that a function $\phi : I \to I$, where $I \subseteq \mathbb{R}$, is *A-convex* if for any I-valued random variable V and event F, we have

$$A[\phi(V) \mid F] \geq \phi(A[V \mid F]) \ .$$

In the case in which $A[\cdot \mid F]$ is a conditional expectation, this is the familiar Jensen inequality. The definition of *A-concave* is obtained by simply reversing the above inequality.

Proposition 1. *Suppose that* $\left\{\succeq_t^F : F \in \mathscr{E}, t \in \mathscr{T}\right\}$ *satisfies event coherence* (*that is*, $A0$ *holds for every* t), *and has a recursive utility representation*, $\left\{U_t^F\right\}$, *with corresponding conditional and intertemporal aggregator families* $\{A[\cdot \mid F]\}$ *and* $\{f_t\}$, *respectively* (*see Definition* 1). *If the function* $f_t(c, \cdot)$ *is A-convex* (*respectively, A-concave*) *for every* $t \in \mathscr{T}$ and $c \in \mathscr{C}$, *then* $\succeq_0^\Omega$ *is information seeking* (*respectively, averse*).

The simple recursive proof of this result is given in Appendix B. A converse of Proposition 1 can be easily formulated, provided we assume a "rich enough" space of acts. For example, in some axiomatizations of additive aggregation that are based on connectedness and continuity (see Skiadas 1997a,b, as well as the references of Section 5) such richness of acts is assumed in order to obtain additivity, and the proof of Proposition 1 can be reversed to obtain a converse. It is also straightforward to formulate strict versions of the notions of information aversion, A-convexity, event coherence, and Proposition 1. We leave the details to the interested reader.

4 Ordinal representation theorems

In this section we formulate conditions on the primitives of Section 2 that guarantee the existence of a recursive utility representation. We give two representation theorems. The first one has the simpler set of assumptions, and derives a recursive utility representation in which conditional aggregators, while sufficiently well-behaved to imply dynamic consistency, need not be monotone on their entire domain. In the second result, monotonicity is obtained at the cost of a more elaborate version of event coherence. The case of additive conditional aggregators is considered in the following section.

In addition to event coherence (A0), we are going to adopt three new assumptions. The following assumption is clearly necessary for any recursive utility representation to exist:

A1. *The preference order* $\succeq_0^\Omega$ *has an (ordinal) utility representation.*

An act, x, is *deterministic* if its corresponding consumption process, $c(x)$, is deterministic (that is, not dependent on its state argument). Deterministic acts will play the role of calibrating devices in our formulation. The assumed properties that enable us to do so are summarized in the following condition:

A2. (*a*) *For any event* F, *and any acts*, x, y, *such that* $x \succ_1^F y$, *there exists a deterministic act* z *such that* $x \succeq_1^F z \succeq_1^F y$.

(*b*) *For any events* F, G, *and any deterministic acts* $x, y \in X_1^F \cap X_1^G$, *we have* $x \succeq_1^F y \Leftrightarrow x \succeq_1^G y$.

The first part of this assumption can be thought of as a weaker version of the requirement that certainty equivalents for time-one preferences exist. The second part is supported by the implicit informal assumptions that the decision maker is indifferent towards information when faced with a deterministic consumption plan, and that the decision maker's enjoyment of deterministic consumption is state-independent.

In order to state our next assumption, we introduce some new notation. For any $F, G \in \mathscr{E}$, $t \in \mathscr{T}$, $x \in X_t^F$, and $y \in X_t^G$, we write $(x, F) \succeq_t (y, G)$ to indicate that, for any deterministic act $z \in X_t^F \cap X_t^G$, we have:

$$(y \succeq_t^G z \Rightarrow x \succeq_t^F z) \quad \text{and} \quad (y \succ_t^G z \Rightarrow x \succ_t^F z) \ .$$

Given assumption A2, the natural interpretation of this condition is that the consequences of x during $\{t, \ldots, T\}$ on event F are at least as desirable as the consequences of y during $\{t, \ldots, T\}$ on event G. Using this notation, we can now formulate our final condition needed to obtain a recursive utility representation:

A3. *For any* $x, y \in X$, $t \in \mathscr{T}$, $F \in \mathscr{F}_t^0(x)$, $G \in \mathscr{F}_t^0(y)$, and $c \in \mathscr{C}$, such that $c_t(x) = c$ on F and $c_t(y) = c$ on G, we have

$$(x, F) \succeq_{t+1} (y, G) \quad \Leftrightarrow \quad (x, F) \succeq_t (y, G) \ ,$$

where, by convention, $(x, F) \succeq_{T+1} (y, G)$ *is always true.*

Condition A3 is a consequence of our implicit informal assumption of time coherence, as discussed in Section 2, and the also informal assumption that the time-t consequences of an act on some event depend only on time-t consumption on the given event and the anticipation of future consequences. To see that, consider the forward implication in A3. By assumption, the time-t consumption on event F under act x coincides with the time-t consumption on event G under act y. Moreover, the statement $(x, F) \succeq_{t+1} (y, G)$ indicates that, from the point of view of time t, anticipated future consequences of act x on event F are at least as desirable as anticipated future consequences of act y on event G. The informal principles referred to above then imply that, relative to time t, the present and anticipated future consequences of act x on event F are at least as desirable as the corresponding consequences of act y on event G. In symbols, $(x, F) \succeq_t (y, G)$. The converse implication follows by similar reasoning, using the strict version of time coherence described in Section 2.

Condition A3 is the basic assumption that ties preferences recursively across time. Its role is analogous to that of Postulate 3 of Koopman's (1960) original formulation for the deterministic case, or the temporal consistency Axiom 3.1 of Kreps and Porteus (1978) in a stochastic setting. What the above interpretation of A3 rules out is that unrealized or past consequences are also among the determinants of present consequences. In Section 6 we discuss ways of relaxing this last informal assumption, while retaining time coherence.

An ordinal representation theorem for recursive utility follows:

Theorem 1. *Suppose that A*0 *holds for* $t = 1$, *and that A*1 *through A*3 *are satisfied. Then the conditional preference family* $\{\succeq_t^F\}$ *has a recursive utility representation.*

The proof of Theorem 1 in Appendix B also shows that the recursive utility representation, $\{U_t^F\}$, of the theorem can be chosen to satisfy

$$(x, F) \succeq_t (y, G) \Leftrightarrow U_t^F(x) \geq U_t^G(y), \qquad F \in \mathscr{F}_t(x), G \in \mathscr{F}_t(y), \tag{3}$$

for all $t \in \{1, \ldots, T\}$. Conversely, the existence of such a recursive utility representation and event coherence imply conditions A1, A2(b), and A3. Condition A2(a) is clearly not necessary, while event coherence is necessary if conditional aggregators are assumed to be (strictly) monotone, a condition that the proof of Theorem 1 does not guarantee.[2] Finally, the assumptions of Theorem 1 imply that event coherence (A0) holds for all $t \in \mathscr{T}$.

We conclude this section with a variant of Theorem 1, in which conditional aggregators are shown to be monotone. This is achieved by replacing event coherence with the following version:

A4. *For any event F and any acts* $x, y \in X_1^F$, *suppose that* $\{F_1^x, \ldots, F_n^x\} \subseteq \mathscr{F}_1(x)$ *and* $\{F_1^y, \ldots, F_m^y\} \subseteq \mathscr{F}_1(y)$ *are both partitions of F. If* $(x, F_i^x) \succeq_1 (y, F_j^y)$ *for all* i, j *such that* $F_i^x \cap F_j^y \neq \emptyset$, *then* $x \succeq_1^F y$.

To interpret A4, one can think of the collection $\{(x, F_i^x) : i = 1, \ldots, n\}$ as a description of the consequences of x on F during the time interval $\{1, \ldots, T\}$, grouped in n scenarios, and analogously for $\{(y, F_i^y) : i = 1, \ldots, m\}$. The statement that $(x, F_i^x) \succeq_1 (y, F_j^y)$ for all i, j such that $F_i^x \cap F_j^y \neq \emptyset$ can then be thought of as expressing the fact that the consequences of x on F during $\{1, \ldots, T\}$ are overall at least as desirable as the corresponding consequences of y, that is, $x \succeq_1^F y$.

A conditional aggregator $A[\cdot \mid F]$ given event F is *monotone* if for all $U, V \in L$, $U \geq V$ on F implies $A[U \mid F] \geq A[V \mid G]$. Under A2, conditions A1, A3 and A4 are clearly necessary for the existence of a recursive utility

[2] Of course, conditional aggregators in Theorem 1 are consistent with event coherence and assumption A3, which together imply that, for any given filtration, preferences over state-contingent consumption plans are dynamically consistent.

representation such that (3) holds, and all the corresponding conditional aggregators of Definition 1 are monotone. Conversely, we have the following version of Theorem 1:

Theorem 2. *If conditions A1 through A4 are satisfied, then the conditional preference family* $\{\succeq_t^F\}$ *has a recursive utility representation that satisfies* (3) *and with corresponding conditional aggregators that are monotone.*

Condition A4 implies weak event coherence for time one preferences, in the sense that part (a) of A0 holds (for $t = 1$, and under A2 and A3, for all t). As a result, conditional aggregators are only shown to be weakly monotone. On the other hand, it is straightforward to formulate strict versions of A4 that deliver event coherence in the sense of A0, or even strict event coherence (meaning that $x \succ_t^F y$ and $x \succeq_t^G y$ implies $x \succ_t^{F \cup G} y$, for disjoint F, G). Such a strict version of A4 would also imply, in the context of Theorem 2, that a conditional aggregator of the form $A[\cdot \mid F]$ is strictly monotone over the set of all random variables of the form $U_t^{\mathscr{F}}(x)$, where $x \in X_t^F$ and $\mathscr{F}$ is any sub-algebra of $\mathscr{F}_t(x)$. The details of these extensions are straightforward, and are left to the interested reader.

5 Incorporating subjective probability

In this section we formulate assumptions that imply a recursive utility representation in which all conditional aggregators take the form of conditional expectations under subjective probabilities.

Let X^0 be the space of all acts under which all uncertainty is resolved at time one:

$$X^0 = \{x \in X : \mathscr{F}_t(x) = 2^\Omega \text{ for all } t \geq 1\} \ .$$

Each element of X^0 can be thought of as a mapping from Ω to $\mathscr{C}^{\mathscr{T}}$, the set of all consumption paths. Time-one preferences over X^0 can therefore be embedded into any of the standard axiomatic settings of subjective expected utility theory. Here we will directly assume such a representation:

A5. *There exists a function* $u : \mathscr{C}^{\mathscr{T}} \to \mathbb{R}$, *and a strictly positive probability* P, *such that, for all* $x, y \in X^0$,

$$x \succeq_1^F y \Leftrightarrow \int_F u(c(x))dP \geq \int_F u(c(y))dP, \qquad F \in \mathscr{E}.$$

Condition A5 can be derived from more elementary assumptions by applying any of a number of available theories of subjective expected utility on finite state-spaces, such as those of Wakker (1989), Nakamura (1990), or Gul (1992) (see also Chew and Karni, 1994; Skiadas, 1997b). The approach of Anscombe and Aumann (1963), or that of Savage (1954) could also be applied after appropriate enlargements of our setting, which are straightforward to implement, and will not be further discussed here.

In place of assumption A2, we are going to assume:

A6. *Given any* $t \in \{1, \ldots, T\}$, $x \in X$, *and* $F \in \mathscr{F}_t(x)$, *there exists some deterministic act* $z \in X^0$ *such that* $z \sim_t^F x$. *If* x *is deterministic, then* z *can be chosen so that it has the additional property* $c(z) = c(x)$.

The proof of the following result can be found in Appendix B.

Theorem 3. *Suppose that conditions A*0, *A*1, *A*3, *A*5, *and A*6 *are all satisfied. Then there exists a recursive utility representation in which all conditional aggregators are conditional expectations relative to the probability P of assumption A*5.

As with Theorems 1 and 2, the recursive representation in Theorem 3 can be selected to satisfy (3). Also, the proof of Theorem 3 uses event coherence only in the following weaker sense: For any $x \in X$, $x^0 \in X^0$, $t \in \{1, \ldots, T\}$, and any disjoint events $F, G \in \mathscr{F}_t(x)$, $x \sim_t^F x^0$ and $x \sim_t^G x^0$ implies $x \sim_t^{F \cup G} x^0$. On the other hand, event coherence is clearly a necessary condition for the theorem's representation.

6 Extensions

We conclude the main part of this paper with a brief discussion of event and time-coherent choice that violates the irrelevance of unrealized alternatives or history independence.

Recursive utility in the sense of Definition 1 satisfies the condition of the irrelevance of unrealized alternatives, meaning that, for every event F and time t, if two acts $x, y \in X_t^F$ are identical on $F \times \{t, \ldots, T\}$ (both in consumption and filtration), then $x \sim_t^F y$. This rules out the possibility that the perceived impact of consumption and information about future consumption depends on past expectations. For example, such dependence can arise if the decision maker expects to feel disappointment or elation as a result of worse or better than expected outcomes. Extending arguments made in Skiadas (1997a,b) in a static setting, we now offer, in outline form, a version of recursive utility that incorporates disappointment aversion.

We adopt the same set of primitives as in Section 2. Since disappointment is impossible with deterministic acts, the latter will play the same role as in the last two sections. In particular, we assume A6, and we adopt the definition of the relation $\succeq_t$ of Section 4. We take as given a utility representation, $\{U_t^F\}$, of $\{\succeq_t^F\}$ that satisfies (3) for all $t \in \mathscr{T}$, and we let $V_t(x) = U_t^{\mathscr{F}_t(x)}(x)$ for all $x \in X$. We also assume that, for some underlying probability, $U_t^F(x) = E[V_t(x) \mid F]$ for all $x \in X_t^F$, and any $F \in \mathscr{E}$ and $t \in \mathscr{T}$. An axiomatic foundation for these assumptions can be based on Skiadas (1997b). (The latter's formulation can be applied to time-t conditional preferences, for every $t \in \{1, \ldots, T\}$, while an additional assumption can be used to ensure that the underlying subjective probability is the same in every period.)

Given the above structure, we now weaken our earlier assumption A3, to incorporate disappointment aversion. Given any filtration $\{\mathscr{F}_t\}$, we define, for convenience, $\mathscr{F}_{-1} = \mathscr{F}_0$.

A7. *For any* $t \in \mathcal{T}$, *and* $x, y \in X$, *suppose that* $F \in \mathcal{F}_t^0$, $F' \in \mathcal{F}_{t-1}^0(x)$, $F \subseteq F'$, and $G \in \mathcal{F}_t^0(y)$, $G' \in \mathcal{F}_{t-1}^0(y)$, $G \subseteq G'$. *If, for some* $c \in \mathcal{C}$, $c_t(x) = c$ *on* F *and* $c_t(y) = c$ *on* G, *then*

$$[(y, G') \succeq_t (x, F') \text{ and } (x, F) \succeq_{t+1} (y, G)] \Rightarrow (x, F) \succeq_t (y, G) \ , \qquad (4)$$

where, by convention, $(y, G') \succeq_{-1} (x, F')$ *and* $(x, F) \succeq_{T+1} (y, G)$ *are always true.*

As with A3, assumption A7 can be thought of in terms of time coherence and an interpretation of consequences. In this case, time-t consequences are allowed to include a sense of disappointment or elation when the arrival of news is worse or better than expected. The condition $(y, G') \succeq_t (x, F')$ in (4) indicates that the expectations formed under y just prior to the arrival of the information that G is true are at least as good as the expectations formed under x just prior to the arrival of the information that F is true. This assumption guarantees that the conclusion $(x, F) \succeq_t (y, G)$ (justified as in A3) would not be reversed because of any feelings of disappointment or elation. In this sense, (4) is consistent with time coherence, but also allows for a sense of disappointment aversion.

One can easily show (in a manner similar to step 4 of the proof of Theorem 1) that A7 is equivalent to the existence of a function $f : \mathcal{T} \times \mathcal{C} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is nonincreasing in its third argument and nondecreasing in its fourth argument, such that

$$V_t(x) = f_t(c_t(x), E[V_t(x) \mid \mathcal{F}_{t-1}(x)], E[V_{t+1}(x) \mid \mathcal{F}_t(x)]), \quad t \in \mathcal{T}, \ x \in X \ . \quad (5)$$

Recursion (5) completely specifies the decision maker's preferences. To see that, define $U_t(x) = E[V_t(x) \mid \mathcal{F}_{t-1}(x)]$ for every $t \in \mathcal{T}$ and $x \in X$, and notice that (5) implies

$$U_t(x) = E[f_t(c_t(x), U_t(x), U_{t+1}(x)) \mid \mathcal{F}_{t-1}(x)], \quad t \in \mathcal{T}, \ x \in X \ . \qquad (6)$$

By the monotonicity of f in its third argument, $U_t(x)$ is uniquely determined by $c_t(x)$ and $U_{t+1}(x)$. Recursion (6) can therefore be used to determine $U_t(x)$ for every time t, which can subsequently be used in (5) to recover $V_t(x)$ for every time t.

Further generalizations can be achieved by weakening A3 or A7 to allow for various forms of history dependence. For example, in the above formulation a sense of disappointment or elation is only assumed to last over a single period. It is left to the interested reader to formulate a generalization of A7 that controls for expectations formed during the complete history. The result is a version of recursion (6) that takes the forward-backward form:

$$U_t(x) = E[f_t(c_t(x), U_0(x), \ldots, U_t(x), U_{t+1}(x)) \mid \mathcal{F}_{t-1}(x)] \ .$$

In this case the whole process $\{U_t(x)\}$ must be determined as a fixed point.

Other types of history dependence can be attributed to the formation of habits. To model such a situation one need only modify A3 by requiring that the complete consumption history prior to t is the same for x on F as for y on G. The result is the same type of recursion as in (2), but now the argument c_t has to be interpreted as the complete history $(c_0, \ldots, c_t)$.

Appendix A: Stochastic differential utility

Duffie and Epstein (1992) introduced a continuous-time version of recursive utility, which they called stochastic differential utility (SDU), but did not discuss the role of the timing of resolution of uncertainty in their setting. The purpose of this section is to extend the definition of SDU to the type of acts introduced in Section 2, but in continuous time, and to obtain a result analogous to Proposition 1, linking the convexity or concavity of an intertemporal aggregator to preferences for information. This appendix is more technical than the rest of the paper, but the intuition is analogous.

We start with a probability space $(\Omega, \mathscr{F}, P)$ (no longer assumed finite), and the bounded real interval $\mathscr{T} = [0, T]$, representing a continuous time horizon. For technical reasons, we assume that the above probability space is *complete* (meaning that every subset of a P-null element of $\mathscr{F}$ is also in $\mathscr{F}$). For our purposes, a *filtration* is defined as any set of σ-algebras of the form $\{\mathscr{F}_t : t \in \mathscr{T}\}$, satisfying the following conditions:

(a) For all $s, t \in \mathscr{T}$ such that $s \leq t$, we have $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$.

(b) $\mathscr{F}_0$ is the trivial σ-algebra (generated by the null events).

(c) For every $t < T$, $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$.

As in the discrete case, a filtration represents a stream of information, where for simplicity we have assumed that at time zero there is always no information.

Consumption bundles are elements of a convex subset, $\mathscr{C}$, of some Euclidean space (or, more generally, of some separable Banach lattice), with $\|\cdot\|$ denoting the norm of $\mathscr{C}$. A *consumption process* is any product-measurable function of the form $c : \Omega \times \mathscr{T} \to \mathscr{C}$, satisfying $E(\int_0^T \|c_t\|^2 dt) < \infty$. As before, we take as primitive a set of acts, X, and we associate with each $x \in X$ a consumption process and a filtration, denoted $c(x)$ and $\{\mathscr{F}_t(x)\}$, respectively. For every $x \in X$, we assume that $c(x)$ is optional[3] relative to the filtration $\{\mathscr{F}_t(x)\}$, which should be interpreted as the restriction that, at each time, present consumption is known.

We define a recursive utility in terms of an *intertemporal aggregator*, that is, a function of the form $f : \mathscr{T} \times \mathscr{C} \times \mathbb{R} \to \mathbb{R}$ satisfying the following technical Lipschitz and growth condition: for some constant K, and all $c \in \mathscr{C}$, $t \in \mathscr{T}$, and $u, v \in \mathbb{R}$, $|f(t, c, u) - f(t, c, v)| \leq K|u - v|$ and $|f(t, c, 0)| \leq K(1 + \|c\|)$. For every $x \in X$, Proposition A1 of Duffie and Epstein (1992) guarantees that there is a unique (up to modifications) square-integrable process[4] $V(x)$ satisfying

$$V_t(x) = E\left[\int_t^T f(s, c_s(x), V_s(x))ds \mid \mathscr{F}_t(x)\right], \qquad t \in \mathscr{T} \ . \tag{7}$$

[3] The consumption process c is *optional* relative to the filtration $\{\mathscr{F}_t\}$ if it is measurable with respect to the σ-algebra on $\Omega \times \mathscr{T}$ generated by the set of RCLL (right continuous and with left limits) consumption processes that are adapted to $\{\mathscr{F}_t\}$, and the Borel σ-algebra on $\mathscr{C}$.

[4] Square-integrability means that $\sup_t V_t(x)$ has finite second moment for all x.

The arguments of Duffie and Epstein (1992) indicate that $V(x)$ can be thought of as a continuous time version of the corresponding process, $V(x)$, in Definition 1. For simplicity, we focus on the time-zero utility $U : X \to \mathbb{R}$, defined by $U(x) = V_0(x)$.

The utility function U is *information seeking* (respectively, *averse*) if for any $x, y \in X$ such that $c(x) = c(y)$ and $\mathscr{F}_t(x) \supseteq \mathscr{F}_t(y)$ for all t, we have $U(x) \geq U(y)$ (respectively, $U(y) \geq U(x)$). In analogy with the discrete-time case, we then have the following result, which is the main contribution of this appendix.

Proposition A. *Suppose that, for every* $(t, c) \in \mathscr{T} \times \mathscr{C}$, $f(t, c, \cdot)$ *is convex (respectively, concave). Then* U *is information seeking (respectively, averse).*

To prove this result, we use the author's following extension of Gronwall's inequality, whose proof is reported by Duffie and Epstein (1992, Lemma B2).

Lemma A. *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space equipped with a filtration*[5] $\{\mathscr{F}_t\}$. *Suppose that* V *is an integrable optional processes,* K *is a constant, and* U *is a product-measurable process. Suppose that the paths of* V *are right continuous almost surely, and that, for every* t, *the paths of the process* $\{E[V_s \mid \mathscr{F}_t] : s \geq t\}$ *are continuous almost surely. If* $V_T \geq 0$ *a.s. and, for all* t, $U_t \geq -K|V_t|$ *and* $V_t = E\left[\int_t^T U_s ds + V_T \mid \mathscr{F}_t\right]$ *a.s., then* $V_t \geq 0$ *a.s. for all* t.

Proof of Proposition A: We assume that f is convex in its last argument; the concave case is analogous. Suppose that $x, y \in X$ satisfy $c(x) = c(y) = c$ and $\mathscr{F}_t(x) \supseteq \mathscr{F}_t(y)$ for all t. Recursion (7) gives

$$E[V_t(x) \mid \mathscr{F}_t(y)] - V_t(y) = E\left[\int_t^T [f_s(c_s, V_s(x)) - f_s(c_s, V_s(y))] ds \mid \mathscr{F}_t(y)\right] .$$

By Jensen's inequality and the Lipschitz condition on f, we have

$$\begin{aligned} E[f_s(c_s, V_s(x)) \mid \mathscr{F}_s(y)] - f_s(c_s, V_s(y)) &\geq f_s(c_s, E[V_s(x) \mid \mathscr{F}_s(y)]) - f_s(c_s, V_s(y)) \\ &\geq -K|E[V_s(x) \mid \mathscr{F}_s(y)] - V_s(y)| \ . \end{aligned}$$

Using a version of Fubini's theorem, the law of iterated expectations, and Lemma A, we find that $E[V_t(x) \mid \mathscr{F}_t(y)] \geq V_t(y)$ for all t, an inequality analogous to (8) in the proof of Proposition 1. In particular, $U(x) \geq U(y)$, completing the proof. □

Applications of Proposition A to asset pricing theory are given by Duffie, Schroder, and Skiadas (1996, 1997).

---

[5] Recall that we have incorporated in the definition of a filtration what are known as the "usual conditions."

Appendix B: Proofs

Proof of Proposition 1

We assume that $f_t(c,\cdot)$ is A-convex; the A-concave case is analogous. Suppose that acts x and y satisfy $x \geq y$ (in the sense of Section 3), and let V be as in Definition 1. We will prove by a backward induction that, for all $t \in \mathscr{T}$,

$$A[V_t(x) \mid \mathscr{F}_t(y)] \geq V_t(y) \ . \tag{8}$$

For $t = 0$, (8) implies $x \succeq_0^\Omega y$, proving that U is information seeking. For $t = T$, (8) holds as an equality, since both sides are equal to $f_T(c_T(x), 0) = f_T(c_T(y), 0)$. Suppose now that $t < T$, and that $A[V_{t+1}(x) \mid \mathscr{F}_{t+1}(y)] \geq V_{t+1}(y)$. This implies that $x \succeq_{t+1}^F y$ for all $F \in \mathscr{F}_{t+1}^0(y)$, and by event coherence (A0), $x \succeq_{t+1}^F y$ for all $F \in \mathscr{F}_t^0(y)$. Therefore, $A[V_{t+1}(x) \mid \mathscr{F}_t(y)] \geq A[V_{t+1}(y) \mid \mathscr{F}_t(y)]$. Using this fact, we then have:

$$\begin{aligned} A[V_t(x) \mid \mathscr{F}_t(y)] &= A[f_t(c_t(x), A[V_{t+1}(x) \mid \mathscr{F}_t(x)]) \mid \mathscr{F}_t(y)] \\ &\geq f_t(c_t(x), A[A[V_{t+1}(x) \mid \mathscr{F}_t(x)] \mid \mathscr{F}_t(y)]) \\ &= f_t(c_t(x), A[V_{t+1}(x) \mid \mathscr{F}_t(y)]) \\ &\geq f_t(c_t(y), A[V_{t+1}(y) \mid \mathscr{F}_t(y)]) = V_t(y). \end{aligned}$$

The first and last equalities follow from the recursive representation (2), the first inequality follows from the A-convexity of f, the second equality follows from the condition in part (a) of Definition 1, and the last inequality follows from the monotonicity of f in its last argument and the inductive hypothesis as indicated above. This shows (8), and completes the proof of Proposition 1. □

Proof of Theorem 1

We proceed in four steps. In the first step we construct an appropriate utility representation for the time-one conditional preference family $\{\succeq_1^F\}$. In the second step we construct a corresponding conditional aggregator family. In the third step, we complete the definition of the utility representation, $\{U_t^F\}$, of the entire conditional preference family $\{\succeq_t^F\}$, extending the construction of step 1. Finally, in the fourth step we construct the functions f_t of recursion (2). Steps 1 and 2 are similar to the proof of Theorem A2 of Skiadas (1997a). The complete argument is presented, however, both for completeness, and because it is needed in other parts of the proof.

Step 1: Given any two sets of acts, X_1 and X_2, we write $X_1 \ll X_2$ to indicate that $X_1 \subseteq X_2$ and for every $F \in \mathscr{E}$ and $x, y \in X_2$ such that $x \succ_1^F y$, there exists a z in X_1 such that $x \succeq_1^F z \succeq_1^F y$. One can easily check that $\ll$ is a transitive relation, a fact we will use shortly. Let now Y be the set of all deterministic acts. Combining assumptions A0, A1, and A3, it follows easily[6] that $\succeq_1^F$ has some utility representation for every $F \in \mathscr{E}$.

By a standard result (see, for example, Fishburn, 1979), there exists a countable $\succeq_1^F$-order dense subset of Y, for every F. Since there are only finitely many events, there exists a countable set $Z \ll Y$. By assumption A2, we also have $Y \ll X$, and therefore $Z \ll X$. We fix such a set of deterministic acts $Z = \{z_1, z_2, \ldots\}$ for the remainder of this proof.

Given any $F \in \mathscr{E}$ and $n \in \{1, 2, 3, \ldots\}$, let the function $d_n^F : X_1^F \to \{0, 1, 2\}$, be defined by

$$d_n^F(x) = \begin{cases} 0 & \text{if } z_n \succ_1^F x; \\ 1 & \text{if } x \sim_1^F z_n; \\ 2 & \text{if } x \succ_1^F z_n \end{cases} .$$

We then define the corresponding utility function $U_1^F : X \to [0, 1]$ through the equations:

$$U_1^F(x) = \sum_{n=1}^{\infty} \frac{d_n^F(x)}{3^n}, \qquad x \in X_1^F \quad .$$

Notice that there is a unique sequence of the form $\{d_n^F(x) : n = 1, 2, \ldots\}$ that corresponds to each value of U_1^F. (The possibility of a double ternary representation of a number in $[0, 1]$ does not create a problem. For suppose we had $U_1^F(x) = U_1^F(y)$, and, for some integer N, $d_N^F(y) = d_N^F(x) + 1$, while $d_n^F(x) = 2$ and $d_n^F(y) = 0$ for all $n > N$. Then one obtains the contradiction $y \succeq_1^F x$ and $x \succ_1^F z_n \succ_1^F y$ for $n > N$.)

Next we show that U_1^F is a utility representation of $\succeq_1^F$. If $x \succeq_1^F y$, then $d_n^F(x) \geq d_n^F(y)$ for every n, and therefore, $U_1^F(x) \geq U_1^F(y)$. If in addition $x \succ_1^F y$, $Z \ll X$ implies that $d_n^F(x) > d_n^F(y)$ for some n, and therefore $U_1^F(x) > U_1^F(y)$. Therefore, $U_1^F(x) \geq U_1^F(y) \Leftrightarrow x \succeq_1^F y$ for all $x, y \in X$.

Step 2: Given the utility representation $\{U_1^F\}$ constructed in step 1, a family of conditional aggregators, $\{A[\cdot \mid F]\}$, can be consistently defined to satisfy (1) for $t = 1$, provided the following condition holds for any $F \in \mathscr{E}$ and $x, y \in X$: For any algebras $\mathscr{F} \subseteq \mathscr{F}_1(x)$ and $\mathscr{G} \subseteq \mathscr{F}_1(y)$, if $F \in \mathscr{F} \cap \mathscr{G}$ and $U_1^{\mathscr{F}}(x) = U_1^{\mathscr{G}}(y)$ on F, then $U_1^F(x) = U_1^F(y)$.

To prove this fact, we define, for any algebra $\mathscr{F} \subseteq \mathscr{F}_1(x)$, $d_n^{\mathscr{F}}(\omega, x) = d_n^F(x)$ whenever $\omega \in F \in \mathscr{F}^0$. Suppose that $U_1^{\mathscr{F}}(x) = U_1^{\mathscr{G}}(y)$ on $F \in \mathscr{F} \cap \mathscr{G}$, for some algebras $\mathscr{F} \subseteq \mathscr{F}_1(x)$ and $\mathscr{G} \subseteq \mathscr{F}_1(y)$. Then $d_n^{\mathscr{F}}(x) = d_n^{\mathscr{G}}(y)$ on F for all n. Fix any n, and for each $i \in \{0, 1, 2\}$, define the set $F_i = \{d_n^{\mathscr{F}}(x) = d_n^{\mathscr{G}}(y) = i\} \cap F$. Since

[6] The idea is to express $\succeq_1^F$ in terms of $\succeq_0^\Omega$, by noticing that if two acts x and y are identical (in consumption and filtration) at time zero and on F^c for all $t \geq 1$, then $x \succeq_0^\Omega y \Leftrightarrow x \succeq_1^F y$. On the other hand, for any two acts x, y that are identical on F for $t \geq 1$, we have $x \sim_1^F y$. The same idea is used in Step 3 below.

$$F_i = \left\{d_n^{\mathscr{F}}(x) = i\right\} \cap F = \left\{d_n^{\mathscr{G}}(y) = i\right\} \cap F \ ,$$

we have $F_i \in \mathscr{F} \cap \mathscr{G}$. Using event coherence (A0), it follows that

$$F_i = \left\{d_n^{\mathscr{F}\cap\mathscr{G}}(x) = i\right\} \cap F = \left\{d_n^{\mathscr{F}\cap\mathscr{G}}(y) = i\right\} \cap F \ .$$

Therefore, $d_n^{\mathscr{F}\cap\mathscr{G}}(x) = d_n^{\mathscr{F}\cap\mathscr{G}}(y)$ on F. Since this is the case for all n, $U_1^{\mathscr{F}\cap\mathscr{G}}(x) = U_1^{\mathscr{F}\cap\mathscr{G}}(y)$ on F. Again by event coherence, it follows that $U_1^F(x) = U_1^F(y)$, proving the consistency property required for conditional aggregators to exist.

Remark: Another property of the above constructed utilities worth mentioning is that, for any algebra $\mathscr{F} \subseteq \mathscr{F}_1(x)$, if $U_1^{\mathscr{F}}(x)$ is constant on $F \in \mathscr{F}$, then $U_1^F(x) = U_1^{\mathscr{F}}(x)$ on F. (To see that, notice that, for any n, if $d_n^{\mathscr{F}}(x)$ is constant on $F \in \mathscr{F} \subseteq \mathscr{F}_t(x)$, then $d_n^F(x) = d_n^{\mathscr{F}}(x)$ on F, an easy consequence of the definitions and event coherence.) As a result, conditional aggregators in the recursive representation can be chosen to have the additional property that for every deterministic U, $A[U \mid F] = U$ for all F.

Step 3: In this step we define the remaining conditional utilities of the representation $\left\{U_t^F\right\}$. We arbitrarily fix some $\bar{c} \in \mathscr{C}$. Given any $x \in X$, and $t \in \{2, \ldots, T\}$, we define the act x_t to satisfy

$$(c_s(x_t), \mathscr{F}_s(x_t)) = \begin{cases} (\bar{c}, \mathscr{F}_t(x)), & \text{if } s < t; \\ (c_s(x), \mathscr{F}_s(x)), & \text{if } s \geq t \ . \end{cases}$$

The utility function U_t^F for $t > 1$ is defined by letting $U_t^F(x) = U_1^F(x_t)$. Finally, we let U_0^Ω be any utility representation of $\succeq_0^\Omega$ (assumed to exist in A1).

Applying A3 (with $F = G$), and using the above utility construction, one can easily check that, for all $t > 1$ and $x, y \in X_t^F$,

$$x \succeq_t^F y \Leftrightarrow x_t \succeq_1^F y_t \ \Leftrightarrow U_1^F(x_t) \geq U_1^F(y_t) \Leftrightarrow \ U_t^F(x) \geq U_t^F(y) \ .$$

This proves that U_t^F is a utility representation of $\succeq_t^F$ for all F and t. It is also immediate from the definitions that (1) holds with the aggregators defined in step 2. This completes the construction of a utility representation, $\left\{U_t^F\right\}$, satisfying (1) for some conditional aggregator family, $\{A[\cdot \mid F]\}$.

Another important property of $\left\{U_t^F\right\}$ that we will need is that (3) holds for all $t \in \{1, \ldots, T\}$. To show that, we first notice that, by A3,

$$(x, F) \succeq_t (y, G) \Leftrightarrow \ (x_t, F) \succeq_1 (y_t, G), \qquad F \in \mathscr{F}_t(x), \ G \in \mathscr{F}_t(y), \ t > 1 \ .$$

It therefore suffices to prove (3) for $t = 1$. To do so, we first notice that, by the construction of $\left\{U_1^F\right\}$ and assumption A2(b), $U_1^F(z) = U_1^G(z)$ for any deterministic $z \in X^F \cap X^G$. Therefore, if $U_1^F(x) \geq U_1^G(y)$, then for every deterministic act z, $U_1^G(y) \geq U_1^G(z)$ implies $U_1^F(x) \geq U_1^F(z)$, and $U_1^G(y) > U_1^G(z)$ implies $U_1^F(x) > U_1^F(z)$. Since U_1^F represents $\succeq_1^F$, we have $(x, F) \succeq_1 (y, G)$. Conversely, suppose that $(x, F) \succeq_1 (y, G)$. It then follows from the definitions in step 1 that $d_n^F(x) \geq d_n^G(y)$ for every n, and therefore $U_1^F(x) \geq U_1^G(y)$. This completes the proof of (3) for all $t \geq 1$.

Step 4: Finally, for every $t \in \mathscr{T}$, we show the existence of a function $f_t : \mathscr{C} \times I_t \to I_t$, $I_t \subseteq \mathbb{R}$, strictly increasing in its last argument, such that

$$V_t(x) = f_t(c_t(x), U_{t+1}^{\mathscr{F}_t(x)}(x)), \quad x \in X, \tag{9}$$

where, by convention, $U_{T+1}^{\mathscr{F}_T(x)}(x) = 0$. Clearly, (9) together with the aggregation result of step 2 gives recursion (2), hence completing the proof of the theorem.

Given any $t \in \mathscr{T}$ and $c \in \mathscr{C}$, let

$$I_t = \left\{ U_{t+1}^{\mathscr{F}_t(x)}(\omega, x) : \omega \in \Omega, x \in X, c_t(\omega, x) = c \right\} .$$

Given our assumptions, it follows easily that I_t is the same for any choice of c. We now claim that (9) consistently defines a strictly increasing function $f_t(c, \cdot)$ on I_t. Suppose that for some $c \in \mathscr{C}$, $t \in \mathscr{T}$, $x, y \in X$, $F \in \mathscr{F}_t^0(x)$, and $G \in \mathscr{F}_t^0(y)$, we have $c_t(x) = c$ on F and $c_t(y) = c$ on G. Then, by A3,

$$U_{t+1}^F(x) \geq U_{t+1}^G(y) \quad \Leftrightarrow \quad U_t^F(x) \geq U_t^G(y) \ ,$$

which confirms our claim, and completes the proof of Theorem 1. □

Proof of Theorem 2

The proof of Theorem 2 is the same as that of Theorem 1, with the exception of step 2 on the construction of conditional aggregators, which should be replaced by the following argument. All utilities are defined as before, and we use the fact that (3) holds. (The proof of (3) is given in step 3 of the proof of Theorem 1.)

Given any event F, let D^F be the set of all random variables of the form $U_1^{\mathscr{F}}(x)$, where $x \in X_1^F$, and $\mathscr{F}$ is a sub-algebra of $\mathscr{F}(x)$. We are first going to define a monotone conditional aggregator $A[\cdot \mid F]$ on D^F. To do so, it suffices to show that for any $x, y \in X_1^F$, and any algebras $\mathscr{F} \subseteq \mathscr{F}(x)$ and $\mathscr{G} \subseteq \mathscr{F}(y)$, $U_1^{\mathscr{F}}(x) \geq U_1^{\mathscr{G}}(y)$ implies that $U_1^F(x) \geq U_1^F(y)$. But because of (3), this is an immediate consequence of assumption A4.

Our next task is to extend the definition of $A[\cdot \mid F]$ to the whole domain L, while preserving monotonicity. Let

$$\overline{D}^F = \left\{ V : V \geq U \text{ on } F, \text{ for some } U \in D^F \right\} .$$

We extend $A[\cdot \mid F]$ to $\overline{D}^F$ by letting

$$A[V \mid F] = \sup\left\{ A[U \mid F] : V \geq U \text{ on } F, U \in D^F \right\} , \quad V \in \overline{D}^F .$$

Finally, we extend $A[\cdot \mid F]$ to the whole of L, by letting

$$A[V \mid F] = \inf\left\{ A[U \mid F] : V \leq U \text{ on } F, U \in \overline{D}^F \right\} , \quad V \in L .$$

The resulting extension $A[\cdot \mid F]$ is finite valued (because utilities are bounded by construction), monotone, and defines a conditional aggregator given F. The proof of Theorem 2 can then be completed just as for Theorem 1. $\square$

Proof of Theorem 3

We begin by defining a utility representation $\{U_t^F\}$, which we will subsequently show to possess the desired properties. We let some $\bar{c} \in \mathscr{C}$ be fixed arbitrarily throughout the proof.

Given any $x \in X$, $t \in \{1, \ldots, T\}$, and $F \in \mathscr{F}_t^0(x)$, we apply A6 to select a deterministic act $z(x, F, t) \in X^0$ satisfying $z(x, F, t) \sim_t^F x$. If x is deterministic, we select $z(x, F, t)$ so that $c(z(x, F, t)) = c(x)$. In terms of $z(x, F, t)$, we define the consumption stream $c(x, F, t) \in \mathscr{C}^{\mathscr{T}}$ by

$$c(x, F, t)_s = \begin{cases} \bar{c}, & \text{if } s < t; \\ c_s(z(x, F, t)), & \text{if } s \geq t. \end{cases}$$

Finally, we define $U_t^F(x) = u(c(x, F, t))$. We extend this definition to arbitrary $F \in \mathscr{F}_t(x)$, by postulating additivity:

$$F \cap G = \emptyset \;\Rightarrow\; U_t^{F \cup G}(x) P(F \cup G) = U_t^F(x) P(F) + U_t^G(x) P(G), \;\; F, G \in \mathscr{F}_t(x) \;\; .$$

This procedure defines a utility function $U_t^F : X_t^F \to \mathbb{R}$, for any $F \in \mathscr{E}$ and $t \in \{1, \ldots, T\}$, which we next show to be a utility representation of $\succeq_t^F$.

Fix any $x, y \in X_t^F$, where $t \geq 1$, and let $\{F_1(x), \ldots, F_n(x)\} \subseteq \mathscr{F}_t^0(x)$ and $\{F_1(y), \ldots, F_m(y)\} \subseteq \mathscr{F}_t^0(y)$ be both partitions of F, corresponding to the events that would be known at time t, in every state of F, under x and y, respectively. Define the act $x^0 \in X^0$ by letting $c(x^0)$ be equal to $\bar{c}$ on F^c, and equal to $c(x, F_i(x), t)$ on F_i for $i \in \{1, \ldots, n\}$. Let y^0 be defined analogously. For every $i \in \{1, \ldots, n\}$, x^0 and $z(x, F_i(x), t)$ are both in X^0 and result in the same consumption on $F_i(x)$. Therefore, by A3 and the definition of $z(x, F_i(x), t)$, we have $x^0 \sim_t^{F_i(x)} z(x, F_i(x), t) \sim_t^{F_i(x)} x$. By event coherence (A0 or the weaker condition of the remark following the statement of Theorem 3), $x \sim_t^F x^0$, and similarly $y \sim_t^F y^0$. Therefore, $x \succeq_t^F y \Leftrightarrow x^0 \succeq_t^F y^0$. Applying A3 again, we have $x^0 \succeq_t^F y^0 \;\Leftrightarrow x^0 \succeq_1^F y^0$, while, by A5 and the definition of $\{U_t^F\}$, $x^0 \succeq_1^F y^0 \;\Leftrightarrow U_t^F(x) \geq U_t^F(y)$. This proves that $\succeq_t^F$ is a utility representation of $\succeq_t^F$. We complete the specification of $\{U_t^F\}$, by letting U_0^Ω be any utility representation of $\succeq_0^\Omega$.

Next we show that equation (3) holds for all $t \in \{1, \ldots, T\}$. The forward implication is an immediate consequence of the facts that $\{U_t^F\}$ is a utility representation of $\{\succeq_t^F\}$, and, by construction, $U_t^F(z) = U_t^G(z)$ for any deterministic act $z \in X_t^F \cap X_t^G$. Conversely, suppose that $(x, F) \succeq_t (y, G)$ and $U_t^G(y) > U_t^F(x)$ for some $F \in \mathscr{F}_t(x)$ and $G \in \mathscr{F}_t(y)$. By A6, there exists some deterministic $z \in X^0$ such that $z \sim_t^G y$, and therefore $U_t^F(z) = U_t^G(z) = U_t^G(y) > U_t^F(x)$. Therefore, while $y \succeq_t^G z$, we have $z \succeq_t^F x$, contradicting the assumption $(x, F) \succeq_t (y, G)$. This proves (3).

The proof of Theorem 3 can now be completed exactly as in step 4 of the proof of Theorem 1. $\square$

References

Anscombe, F., Aumann, R.: A definition of subjective probability. Annals of Mathematical Statistics **34**, 199–205 (1963)

Chew, S. H., Epstein, L.: The structure of preferences and attitudes towards the timing of resolution of uncertainty. Int. Econ. Rev. **30**, 103–117 (1989)

Chew, S. H., Epstein, L.: Recursive utility under uncertainty. In: A. Kahn, N. Yannelis (eds.) Equilibrium theory with an infinite number of commodities. New York: Springer 1991

Chew, S. H., Karni, E.: Choquet expected utility with a finite state space: Commutativity and act independence. Journal of Economic Theory **62**, 469–479 (1994)

Dreze, J., Modigliani, F.: Consumption decisions under uncertainty. Journal of Economic Theory **5**, 165–177 (1972)

Duffie, D., Epstein, L.: Stochastic differential utility. Econometrica **60**, 353-394 (appendix with C. Skiadas) (1992)

Duffie, D., Schroder, M., Skiadas, C.: Recursive valuation of defaultable securities and the timing of resolution of uncertainty. Annals of Applied Probability **6**, 1075–1090 (1996)

Duffie, D., Schroder, M., Skiadas, C.: A term structure model with preferences for the timing of resolution of uncertainty. Economic Theory **9**, 3–22 (1997)

Epstein, L.: Behavior under risk: Recent developments in theory and applications. In: J.-J. Laffont (ed.) Advances in economic theory: Sixth World Congress, Cambridge, UK: Cambridge University Press 1992

Fishburn, P.: Utility theory for decision making. Huntington, New York: Krieger 1979

Grant, S., Kajii, A., Polak, B.: Preference for information. Working paper (1996)

Gul, F.: Savage's Theorem with a finite number of states. Journal of Economic Theory **57**, 99–110 (1992)

Johnsen T., Donaldson, J.: The structure of intertemporal preferences under uncertainty and time consistent plans. Econometrica **53**, 1451–1458 (1985)

Karni, E., Schmeidler D.: Utility theory with uncertainty. In: W. Hildenbrand, H. Sonnenschein (eds.) Handbook of mathematical economics, Vol. 4. Amsterdam: North Holland 1990

Koopmans, T.: Stationary ordinal utility and impatience. Econometrica **28**, 287–309 (1960)

Kreps, D., Porteus, E.: Temporal resolution of uncertainty and dynamic choice theory. Econometrica **46**, 185–200 (1978)

Mossin, J.: A note on uncertainty and preferences in a temporal context. American Economic Review **59**, 172–173 (1969)

Nakamura, Y.: Subjective expected utility with non-additive probabilities in finite state spaces. Journal of Economic Theory **51**, 346–366 (1990)

Savage, L.: The foundations of statistics. New York: Dover Publications 1954

Spence, M., Zeckhauser, R.: The effect of the timing of consumption decisions and the resolution of lotteries on the choice of lotteries. Econometrica **40**, 401–403 (1972)

Skiadas, C.: Conditioning and aggregation of preferences. Econometrica **65**, 347–367 (1997a)

Skiadas, C.: Subjective probability under additive aggregation of conditional preferences. Journal of Economic Theory (forthcoming) (1997b)

Wakker, P.: Additive representation of preferences, a new foundation of decision analysis. Dordrecht: Kluwer Academic Publishers 1989