Affiliation, Equilibrium Existence and Revenue Ranking of Auctions*

Luciano I. de Castro †

December 2010

Abstract

Affiliation has been a prominent assumption in the study of economic models with statistical dependence. Despite its large number of applications, especially in auction theory, affiliation has limitations that are important to be aware of. This paper shows that affiliation is a restrictive condition and the intuition usually given for its adoption may be misleading. Also, other usual justifications for affiliation are not compelling. Moreover, some implications of affiliation—namely, equilibrium existence in first-price auctions and the revenue dominance of second-price auctions—do not generalize to other definitions of positive dependence.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: affiliation, positive dependence, statistical dependence of types, conditional independence, de Finetti's theorem, minimally informative random variable, auctions, pure strategy equilibrium, revenue ranking.

*First version of this paper: de Castro (2007). I am grateful to Nabil Al-Najjar, Aloisio Araujo, Alain Chateauneuf, Maria Ángeles de Frutos, Juan Dubra, Ángel Hernando, Vijay Krishna, Andreu Mas-Colell, Steven Matthews, Flavio Menezes, Paul Milgrom, Benny Moldovanu, Paulo K. Monteiro, Humberto Moreira, Stephen Morris, Sergio Parreiras, Nicola Persico, Jeroen Swinkels, Steve Tadelis, Steven Williams and Charles Zheng for helpful conversations.

†Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston IL 60208. E-mail: decastro.luciano@gmail.com.

1 Introduction

Asymmetric information is a central theme in modern economics, not only in game theory, but also in industrial organization, general equilibrium, group decision, finance and many other subdisciplines. Most models assume that each agent privately knows a random variable, and these random variables are statistically independent. Although independence is convenient for theoretical manipulations, it is considered a restrictive and unrealistic assumption. Independence is regarded as restrictive because it is satisfied by a "knife-edge" set of distributions, and unrealistic because there are many potential sources of correlation in the real world: media, education, culture or even evolution. Perceiving these limitations early on, economists tried to surpass the mathematical difficulties and include statistical dependence in their models.

The introduction of affiliation was a milestone in the study of dependence in economics. This remarkable contribution was made by Milgrom and Weber (1982a), who borrowed a statistical concept (multivariate total positivity of order 2, MTP_2) and applied it to a general model of symmetric auctions.[1] Affiliation is a generalization of independence—see its definition in section 2—that was introduced through the appealing *positive dependence intuition*: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely" (Milgrom and Weber (1982a, p.1096)). Among many important results, Milgrom and Weber (1982a), were able to show that positive dependence (in the form of affiliation) does not create problem for *pure strategy equilibrium existence*,[2] but it affects in a clear way the *revenue rank-*

[1]In two previous papers, Milgrom (1981b) and Milgrom (1981a) presented results that used a particular version of the concept, under the name "monotone likelihood ratio property" (MLRP). It is also clear that Wilson (1969) and Wilson (1977) influenced the development of the affiliation idea. Nevertheless, the concept was fully developed and the term affiliation first appeared in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b). When there is a density function, the property had been previously studied by statisticians under different names. Lehmann (1966)) calls it Positive Likelihood Ratio Dependence (PLRD), Karlin (1968) calls it Total Positivity of order 2 (TP_2) for the case of two variables or Multivariate Total Positivity of Order 2 (MTP_2) for the multivariate case.

[2]Although equilibria in mixed strategies always exist (Jackson and Swinkels (2005)), first-price auctions may fail to possess a pure strategy equilibrium when types are dependent. However, Milgrom and Weber (1982a) proved that affiliation ensures the existence of a symmetric monotonic (increasing) pure strategy equilibrium (SMPSE) for symmetric first-price auctions. Milgrom and Weber (1982a) also proved the existence of equilibrium for second-price auctions with interdependent values. In our setup (private values), the second-price auction always has an equilibrium in weakly dominant pure strategies, which simply consists of bidding the private value.

ing of auctions, that is, under affiliation, the English and the second-price auction give higher expected revenue than the first-price auction (in self-explanatory symbols: $R_E \geqslant R_2 \geqslant R_1$).[3] These two results suggest an economic interpretation in terms of comparative statics: when the assumption of independence is relaxed in the direction of positive dependence, equilibrium is not a problem and the revenue superiority of the English auction (and second-price auction) increases. From an economic point of view this comparative statics exercise is very interesting, since it clearly indicates what happens to the conclusion of the revenue equivalence theorem (RET) when one of its assumptions is relaxed (from independence to affiliation).[4]

For a quarter of a century, affiliation has been part of the foundations of auction theory and almost synonymous with dependence in auctions. Affiliation's monotonicity properties (see Theorem 5 of Milgrom and Weber (1982a)) combine well with natural properties of auctions, simplifying the analysis and allowing useful predictions. But the success of affiliation is not restricted to auction theory. Whenever information is important, affiliation may potentially be applied. In fact, researchers in many different areas of economics and finance used the concept to obtain useful results.[5] In sum, few theoretical tools achieved as broad an impact as affiliation.

However, as with any scientific achievement, affiliation has limitations. The purpose of this paper is to assess affiliation (and its implications) as an enduring foundation for a theory of dependence in economics.

The first step of our analysis is the observation that Milgrom and Weber were mainly interested in the setting of positive dependence of bidders' valuations. We recall that many different notions of positive dependence are available in the literature, but their rank is eventually not clear or sometimes even wrongly stated. Section 3 defines seven different notions of positive dependence and establishes

[3]For private value auctions, which is the focus of this paper, English and second-price auctions are equivalent, which implies $R_E = R_2$. See Milgrom and Weber (1982a).

[4]Besides independence, the RET requires other restrictive conditions, such as symmetry and risk neutrality. The revenue ranking of auctions is undetermined if all those assumptions are relaxed. Thus, the importance of the result is akin to a comparative statistics exercise: holding everything else fixed, what changes if independence is relaxed in the direction of positive dependence?

[5]For instance, Bergin (2001) used affiliation to obtain a generalization of a theorem by Aumann (1976) for the aggregation of information by a set of individuals; Persico (2000) proved a theorem about the usefulness of information for a decision maker under affiliation; and Sobel (2006) also used affiliation to study aggregation of information by groups. This list represents just a very small sample of papers; it would be almost impossible to cite all applications.

a clear rank among them. It turns out that affiliation is the most restrictive one. From this result, a natural question is: can Milgrom and Weber's results be extended to weaker notions?

Section 4 focus on two of the main implications of affiliation—namely, equilibrium existence and the revenue ranking between the first and second price auctions—and show that these implications can be slightly generalized in the particular case of private values, but not much. Specifically, theorems 4.1 and 4.2 present the first counterexamples of the failure of equilibrium existence and Milgrom and Weber's revenue ranking, respectively, under a familiar yet strong concept of positive dependence: first-order stochastic dominance.

These results suggest concerns about the applicability of the theory. If affiliation's implications do not generalize, it is extremely important to verify whether affiliation is typical or not. This is the theme of section 5. Subsection 5.1 shows that affiliation is a very restrictive condition, that is, it shows that the set of affiliated density functions is a meager set (the complement is open and dense in the standard topology of continuous functions). Subsection 5.2 considers one of the most used justifications for affiliation, namely, conditional independence and shows that this justification works only in special cases. Subsection 5.3 discusses the justifications for the use of affiliation in other sciences.

From this, section 6 briefly reviews the theoretical, experimental and empirical literature on affiliation and observes that there is little support for affiliation on its theory. All these observations suggest that we need novel approaches for dealing with dependence in economics. Section 7 concludes with observations in this direction. An appendix collects the proofs.

Although affiliation has a broad scope of application now, we will consider mainly auctions as the benchmark field of interest. Thus, we begin in the next section by describing the standard auction model.

2 Basic model and definitions

As emphasized above, our main results are not restricted to auctions. However, since affiliation's main implications discussed in section 4 refer to auctions, we will describe an auction model below.

There are n bidders, $i = 1, ..., n$. Bidder i receives private information $t_i \in [\underline{t}, \bar{t}]$ which is the value of the object for himself. The usual notation $t = (t_i, t_{-i})$ = $(t_1, ..., t_n) \in [\underline{t}, \bar{t}]^n$ is adopted. The (private) values are distributed according to a pdf $f : \left[\underline{t}, \overline{t}\right]^n \to \mathbb{R}_+$ which is symmetric. That is, if $\pi : \{1, ..., n\} \to \{1, ..., n\}$ is a permutation, $f(t_1, ..., t_n) = f\left(t_{\pi(1)}, ..., t_{\pi(n)}\right)$. Let $\overline{f}(x) = \int f(x, t_{-i})\, dt_{-i}$ be a marginal of f. Our main interest is the case where f *is not* the product of its marginals, that is, the case where the types are dependent. We denote by $f(t_{-i} \mid t_i)$ the conditional density $f(t_i, t_{-i}) / \overline{f}(t_i)$.

After knowing his value, bidder i places a bid $b_i \in \mathbb{R}_+$. He receives the object if $b_i > \max_{j \neq i} b_j$. We consider both first and second-price auctions with *private values*. This means that the private information of each bidder (type) is also that bidder's value for the object. As Milgrom and Weber (1982a) show, second-price and English auctions are equivalent in the case of private values, as we assume here. In a first-price auction, if $b_i > \max_{j \neq i} b_j$, bidder i's utility is $u(t_i - b_i)$ and is $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. In a second-price auction, bidder i's utility is $u(t_i - \max_{j \neq i} b_j)$ if $b_i > \max_{j \neq i} b_j$ and $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. For both auctions, ties are randomly broken.

A pure strategy is a function $b : [0, 1] \to \mathbb{R}_+$, which specifies the bid $b(t_i)$ for each type t_i. The interim payoff of bidder i, who bids β when his opponent $j \neq i$ follows $b : [0, 1] \to \mathbb{R}_+$ is given by

$$\Pi_i(t_i, \beta, b(\cdot)) = u(t_i - \beta)\, F\left(b^{-1}(\beta) \mid t_i\right) = u(t_i - \beta) \int_{\underline{t}}^{b^{-1}(\beta)} f(t_j \mid t_i)\, dt_j,$$

if it is a first-price auction and

$$\Pi_i(t_i, \beta, b(\cdot)) = \int_{\underline{t}}^{b^{-1}(\beta)} u(t_i - b(t_j))\, f(t_j \mid t_i)\, dt_j,$$

if it is a second-price auction.

We focus attention on symmetric monotonic pure strategy equilibrium (SMPSE), which is defined as $b(\cdot)$ such that $\Pi_i(t_i, b(t_i), b(\cdot)) \geqslant \Pi_i(t_i, \beta, b(\cdot))$ for all β and t_i. The usual definition requires this inequality to be true only for almost all t_i. This stronger definition creates no problems and makes some statements simpler, such as those about the differentiability and continuity of the equilibrium bidding function (otherwise, such properties should be qualified by the expression "almost everywhere"). Finally, under our assumptions, the second price auction always has a SMPSE in a weakly dominant strategy, which is $b(t_i) = t_i$.

By reparametrization, we may assume, without loss of generality, $\left[\underline{t}, \overline{t}\right] = [0, 1]$. It is also useful to assume $n = 2$, but this is not necessary for most of the results. We also assume risk neutrality, i.e., $u(x) = x$. Thus, unless otherwise stated, the results will be presented under the following setup:

BASIC SETUP: *There are $n = 2$ risk neutrals bidders ($u(x) = x$), with private values distributed according to a symmetric density function $f : [0,1]^2 \to \mathbb{R}_+$.*

Affiliation is formally defined as follows.[6]

Definition 2.1 *The density function $f : \left[\underline{t}, \overline{t}\right]^n \to \mathbb{R}_+$ is affiliated if $f(t) f(t') \leqslant f(t \wedge t') f(t \vee t')$, where $t \wedge t' = (\min\{t_1, t_1'\}, ..., \min\{t_n, t_n'\})$ and $t \vee t' = (\max\{t_1, t_1'\}, ..., \max\{t_n, t_n'\})$.*

It is useful to introduce the following notation: $\mathcal{D}$ will denote the set of all densities:

$$\mathcal{D} \equiv \{f : [0,1]^n \to \mathbb{R}_+ : \int_{[0,1]^n} f(t)dt = 1\}.$$

The set of all continuous densities will be denoted $\mathcal{C}$ and $\mathcal{A}$ will denote the set of affiliated (continuous or not) densities.

3 Positive dependence notions

Affiliation was introduced through the *positive dependence intuition*: "a high value of one bidder's estimate makes high values of the others' estimates more likely" (Milgrom and Weber (1982a, p. 1096)). This intuition is very appealing, because positive dependence describes a circumstance likely to happen in the real world. In fact, many authors introduce affiliation through this intuition or some of its variations.

Affiliation captures this intuition, as we illustrate in Figure 1, below. Affiliation requires that the product of weights at points (x', y') and (x, y) (where both values are high or both are low) be greater than the product of weights at (x, y') and (x', y) (where they are high and low, alternatively). In other words, the distribution puts more weight on the points in the diagonal than outside it.

[6] Affiliation is equivalent to MLRP in the particular case of two variables with density function. It is possible to define affiliation even if the joint distribution has no density function. See Milgrom and Weber (1982a).

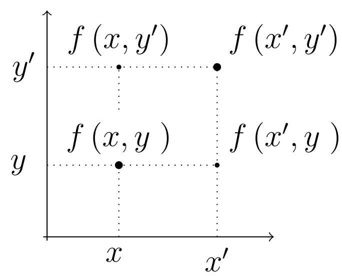

Figure 1 — The pdf f is affiliated if $x \leqslant x'$ and $y \leqslant y'$ imply $f(x, y') f(x', y) \leqslant f(x', y') f(x, y)$.

However, as long as we are interested in *positive dependence*, as this intuition suggests, affiliation is not the only definition available. In the statistical literature many concepts have been proposed to correspond to the notion of positive dependence. For simplicity, let us consider only the bivariate case and assume that the two real random variables X and Y have joint distribution F and strictly positive density function f. The following concepts are formalizations of the notion of positive dependence for X and Y:[7]

Property I — X and Y are positively correlated (PC) if $cov(X, Y) \geqslant 0$.

Property II — X and Y are said to be positively quadrant dependent (PQD) if for all non-decreasing functions g and h, $cov(g(X), h(Y)) \geqslant 0$.

Property III — The real random variables X and Y are said to be associated (As) if for all non-decreasing functions g and h, $cov(g(X, Y), h(X, Y)) \geqslant 0$.

Property IV — Y is said to be left-tail decreasing in X (denoted LTD($Y|X$)) if for all y, the function $x \mapsto \Pr[Y \leqslant y | X \leqslant x]$ is non-increasing in x. X and Y satisfy Property IV if LTD($Y|X$) and LTD($X|Y$).

Property V — Y is said to be positively regression dependent on X (denoted PRD($Y|X$)) if $\Pr[Y \leqslant y | X = x] = F(y|x)$ is non-increasing in x for all y. X and Y satisfy Property V if PRD($Y|X$) and PRD($X|Y$).[8]

[7] Most of the concepts can be properly generalized to multivariate distributions. See, for example, Lehmann (1966) and Esary, Proschan, and Walkup (1967). The hypothesis of strictly positive density function is made only for simplicity.

[8] This property is also known as monotonicity in the first-order stochastic dominance sense.

Property VI — Y is said to be Inverse Hazard Rate Decreasing in X (denoted IHRD($Y|X$)) if $\frac{F(y|x)}{f(y|x)}$ is non-increasing in x for all y, where $f(y|x)$ is the pdf of Y conditional to X. X and Y satisfy Property VI if IHRD($Y|X$) and IHRD($X|Y$).

Since there are many alternative definitions of positive dependence, a natural question is: "How do such definitions compare with affiliation?" The following theorem provides the answer.[9]

Theorem 3.1 *Let affiliation be Property VII. Then,*

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I),$$

and all implications are strict.

The main contribution of this theorem is around properties VII (affiliation), VI and V (first-order stochastic dominance), which are the most usual properties in economics. While Milgrom and Weber have proved $VII \Rightarrow VI$, we were not able to find a reference for the implication $VI \Rightarrow V$. The counterexamples $VI \not\Rightarrow VII$ and $V \not\Rightarrow VI$ are also important, since some confusion may arise with respect to the ranking of these properties.[10] Moreover, we are not aware of such counter-examples in the literature.

This theorem also sets the stage for our main problem. Since affiliation is just the strongest positive dependence property, could Milgrom and Weber choose a weaker property to get their results? As we shall see, the answer will be in the negative.

4 Affiliation's implications and positive dependence

Affiliation has been used in the proof of many results. These results can be classified in two groups: facts that are already true for the independent case (affiliation

[9]For this theorem, we used only seven concepts for simplicity. Yanagimoto (1972) defines more than thirty concepts of positive dependence and, again, affiliation is the most restrictive of all but one.

[10] A casual reader may think that Milgrom (1981a, Proposition 1) states that V is equivalent to VII and Riley (1988, Lemma 1) claims that a strict version of properties IV and V implies property VI. However, this is not the case—the mentioned results are formally correct. Theorem 3.1 helps to appreciate the subtle aspect of their claims.

allows a generalization) and predictions that are qualitatively different from the case of independence. In this section, we will focus on one implication for each of these groups.

The first one is the existence of symmetric monotonic pure strategy equilibrium (SMPSE) for first price auctions, generalized from independence to affiliation. The second one is the revenue ranking of auctions: under affiliation, the English and the second-price auction give expected revenue at least as high as the first price auction (a fact that we denote by $R_2 \geqslant R_1$). This last result is in contrast with the case of independence, where the Revenue Equivalence Theorem (RET) implies the equality of the expected revenues $(R_2 = R_1)$.[11,12] Both implications were obtained by Milgrom and Weber (1982a) and I chose them because of their importance. The purpose of this section is to verify whether these implications (existence of SMPSE and $R_2 \geqslant R_1$) are true in a more general setting.

4.1 Equilibrium existence

Is the existence of SMPSE true under other definitions of positive dependence (see section 3)? Theorem 4.1 below shows that the following property is sufficient:[13]

Property VI′ — The joint (symmetric) distribution of X and Y satisfy Property VI′ if for all x, x' and y in $[0, 1]$,

$$x \geqslant y \geqslant x' \Rightarrow \frac{F(y|x')}{f(y|x')} \geqslant \frac{F(y|y)}{f(y|y)} \geqslant \frac{F(y|x)}{f(y|x)}.$$

It is easy to see that Property VI implies Property VI′ (under symmetry and full support). Thus, the question becomes whether or not it is possible to generalize the existence of SMPSE for Property V or even further.

If we define $\Pi(x, y) = (x - b(y)) F(y|x)$, where $b(\cdot)$ is a candidate for symmetric equilibrium,[14] then equilibrium existence is equivalent to $\Pi(x, x) \geqslant \Pi(x, y)$. Since $b(\cdot)$ is monotonic, one may conjecture that the monotonicity of

[11] Since affiliation contains independence as a special case, the results can be *qualitatively* different, but must have a logic overlap.

[12] Both the revenue ranking under affiliation and the RET requires symmetry, risk neutrality and the same payoff by the lowest type of bidders.

[13] Motivated by an earlier version of this paper, Monteiro and Moreira (2006) obtained other generalizations of equilibrium existence for non-affiliated variables. Their results are not directly related to positive dependence properties.

[14] This candidate is increasing and unique, as we can show using standard arguments. See Maskin and Riley (1984) or de Castro (2008).

$F(y|x)$ — as Property V assumes — may be sufficient for equilibrium existence, through some single crossing arguments (as in Athey (2001)). Since Property V is still a strong property of positive dependence, this conjecture may be considered reasonable. In fact, the reader may think that the following recent result by van Zandt and Vives (2007) actually *proves* that first-order stochastic dominance is sufficient for equilibrium existence in auctions:

Theorem (van Zandt and Vives, 2007): *Assume that for each player i:*

1. *the utility function is supermodular in the own player's action a_i, has increasing differences in (a_i, a_{-i}), and has increasing differences in (a_i, t); and*

2. *the beliefs mapping $p_i : T_i \to \mathcal{M}_i$ is increasing in the first-order stochastic dominance partial order.*

Then there exist a greatest and a least Bayesian Nash equilibrium, and each one is in monotone strategies.

Despite these compelling reasons, the conjecture that Property V is sufficient for equilibrium existence in auctions is actually false; the following theorem clarifies that SMPSE existence does not generalize beyond Property VI'.[15]

Theorem 4.1 *If $f : [0,1]^2 \to \mathbb{R}$ satisfies Property VI′, there is a SMPSE. Nevertheless, Property V is not sufficient for the existence of SMPSE.*

This theorem shows that the rationale for existence of SMPSE existence for Property V do not survive a formalization of the result. Simply, Property V is not strong enough to control the equilibrium inequality $\Pi(x,x) \geqslant \Pi(x,y)$ for every pair of points (x,y).

4.2 Revenue ranking

The next implication—$R_2 \geqslant R_1$—is also an inequality, but it is an inequality over expected values, not specific realizations. For some realizations of the variables,

[15] van Zandt and Vives (2007)'s main result does not apply because even simple auctions with 2 players and private-values do not satisfy one of their assumptions (increasing differences). In fact, if $t_i > a'_j > a'_i > a_j > a_i$ then $(t_i - a'_i)1_{[a'_i > a'_j]} - (t_i - a'_i)1_{[a'_i > a_j]} = -(t_i - a'_i) < 0$ while $(t_i - a_i)1_{[a_i > a'_j]} - (t_i - a_i)1_{[a_i > a_j]} = 0$, to the contrary of the increasing differences requirement.

the second-price auction can give less revenue than the first-price auction, but for the inequality $R_2 \geqslant R_1$ to be true is sufficient that the opposite happens on *average*. Since this is a statement about average cases, one could expect that the revenue ranking $R_2 \geqslant R_1$ would be stable across the cases of positive dependence.

There is yet another way of reaching the same conclusion: it is the intuition for the revenue ranking $R_2 \geqslant R_1$, which is a contribution of Klemperer (2004, p.48-9):

> [In a first-price auction, a] player with value $v + dv$ who makes the same bid as a player with a value of v will pay the same price as a player with a value of v when she wins, but because of affiliation she will expect to win a bit less often [than in the case of independence]. That is, her higher signal makes her think her competitors are also likely to have higher signals, which is bad for her expected profits.
>
> But things are even worse in a second-price affiliated private-values auction for the buyer. Not only does her probability of winning diminish, as in the first-price auction, but her costs per victory are higher. This is because affiliation implies that contingent on her winning the auction, the higher her value the higher expected second-highest value which is the price she has to pay. Because the person with the highest value will win in either type of auction they are both equally efficient, and therefore the higher consumer surplus in first-price auction implies higher seller revenue in the second-price auction.

This intuition appeals mainly to the notion of positive dependence. Thus, the intuition should lead us to believe that the revenue ranking is still valid under weaker forms of positive dependence. Despite these intuitive arguments, however, the following theorem shows that the implication $R_2 \geqslant R_1$ is not robust for other definitions of positive dependence.

Theorem 4.2 *If f satisfies Property VI' (see definition above), then the second-price auction gives greater revenue than the first-price auction $(R_2 \geqslant R_1)$. Specifically, the revenue difference is given by*

$$n \int_0^1 \int_0^x b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where n is the number of players and $b(\cdot)$ is the first-price equilibrium bidding function, or by

$$n \int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \quad (1)$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$.

More importantly, Property V is not *sufficient for this revenue ranking.*

These results suggest that affiliation's implications are not robust. This would not be a reason for concern, however, if affiliation were indeed typical. It is natural, therefore, to examine more closely the settings where affiliation could be expected to hold. The following section makes some comments about this issue.

5 Remarks about affiliation's representativeness

This section evaluates how typical affiliation is from mathematical, economical and methodological points of view. Subsection 5.1 considers a mathematical notion of smallness and show that the set of affiliated density functions satisfies it; in other words, affiliation is a restrictive condition. A common justification for affiliation in economics is conditional independence, discussed in subsection 5.2. Since other sciences use affiliation, subsection 5.3 considers the methodological reasons for this use and observes that they do not carry over into economics.

5.1 Affiliation holds in non-generic settings

In this section, we show that the set of affiliated densities is small in the set of continuous densities. There are two ways to characterize a set as small: topological and measure-theoretic. Although it is possible to show that affiliation is restrictive in the measure-theoretic sense (see an earlier version of this paper, de Castro (2007)), here we limit ourselves to the topological result, which is simpler.

Recall that $\mathcal{C}$ denotes the set of continuous density functions $f : [0,1]^n \to \mathbb{R}_+$ and $\mathcal{A}$, the set of affiliated densities (continuous or not). Endow $\mathcal{C}$ with the standard topology for the set of continuous functions, that is, the topology defined by the norm of the sup:

$$\|f\| = \sup_{x \in [0,1]^n} |f(x)|.$$

The following theorem shows that the set of continuous affiliated densities is small in the topological sense. The proofs of this and of all other results are given in the appendix.

Theorem 5.1 *The set of continuous affiliated density functions $\mathcal{C} \cap \mathcal{A}$ is meager.*[16] *More precisely, the set $\mathcal{C} \backslash \mathcal{A}$ is open and dense in $\mathcal{C}$.*

The proof of this theorem is given in the appendix, but the main idea is simple. To prove that $\mathcal{C} \backslash \mathcal{A}$ is open, we take a pdf $f \in \mathcal{C} \backslash \mathcal{A}$ which does not satisfy the affiliated inequality for some points t, $t' \in [0,1]^2$, that is, $f(t) f(t') > f(t \wedge t') f(t \vee t') + \eta$, for some $\eta > 0$. By using such η, we can show that for a function g sufficiently close to f, the above inequality is still valid, that is, $g(t) g(t') > g(t \wedge t') g(t \vee t')$ and thus is not affiliated. To prove that $\mathcal{C} \backslash \mathcal{A}$ is dense, we choose a small neighborhood V of a point $\hat{t} \in [0,1]^2$, such that for all $t \in V$, $f(t)$ is sufficiently close to $f\left(\hat{t}\right)$. This can be done because f is continuous. We then perturb the function in this neighborhood to maintain the failure of the affiliation inequality.

Maybe more instructive than the proof is the understanding of why the result is true: simply, affiliation requires an inequality to be satisfied everywhere (or almost everywhere). This is a strong requirement, and it is the source of affiliation's restrictiveness.

Although the restrictiveness of affiliation seems to be a "folk theorem," it was never stated or formally proven. Note that the conclusion of Theorem 5.1 may be sensitive to the space and norm considered, that is, the result might be false without the "right" statement, which indicates the value of formalizing it. Thus, Theorem 5.1 (together with the measure theoretical result proved in de Castro (2007)) fill this gap in the literature.

5.2 Conditional independence

A standard way to justify affiliation is to appeal to conditional independence. In fact, affiliation was originally motivated using conditional probabilities (see Milgrom and Weber (1982a, p. 1094). Conditional independence models assume that the signals of bidders are conditionally independent, given a variable v (the

[16]A meager set (or set of first category) is the union of countably many nowhere dense sets, while a set is nowhere dense if its closure has an empty interior. Thus, the theorem says more than that $\mathcal{C} \cap \mathcal{A}$ is meager: $\mathcal{C} \cap \mathcal{A}$ is itself a nowhere dense set, according to the second claim in the theorem.

intrinsic value of the object, for instance). Since symmetry is the same as exchangeability, which is the main assumption of de Finetti's Theorem, some auction specialists seem to believe that de Finetti's Theorem implies that conditional independence holds in symmetric auctions *without loss of generality*. De Finetti's theorem states the following:

De Finetti's Theorem. *Consider a sequence of random variables X_1, X_2,..., and assume that they are exchangeable, that is, assume that the distribution of $(X_1, ..., X_n)$ is equal to the distribution of $\left(X_{\pi(1)}, ..., X_{\pi(n)}\right)$, for any n and any permutation $\pi : \mathbb{N} \to \mathbb{N}$. Then, there is a random variable Q such that all X_1, X_2,..., are conditionally independent (and identically distributed) given Q.*[17]

Unfortunately, however, de Finetti's theorem *is not valid* for standard models of auction theory, even assuming symmetry. The reason is that standard auction models consider a finite number of players and, hence, a finite number of random variables. De Finetti's theorem is valid only for an (infinite) sequence of random variables.[18] The following example illustrates the problem:

Example 5.2 *Consider two random variables, X_1 and X_2, taking values in $\{0, 1\}$, with joint distribution given by: $P(X_1 = 0, X_2 = 1) = P(X_1 = 1, X_2 = 0) = \frac{1}{2} - \varepsilon$ and $P(X_1 = 0, X_2 = 0) = P(X_1 = 1, X_2 = 1) = \varepsilon$. It is easy to see that X_1 and X_2 are symmetric (exchangeable). In the appendix, we show that the conclusion of de Finetti's Theorem cannot hold if $\varepsilon < 1/4$.*[19]

Thus, de Finetti's Theorem *does not* imply that conditional independence is a generic condition in symmetric auctions. However, even if we are ready to assume conditional independence, this is not yet sufficient for affiliation. To see

[17] De Finetti proved this theorem for the case where the X_i are Bernoulli variables. Hewitt and Savage (1955) extended it to the general setting. The statement above is somewhat vague. A precise statement is as follows: Let X_1, X_2,..., be an exchangeable sequence of random variables with values in a set S. Then there exists a probability measure μ on the set of probability measures $\Delta(S)$ such that for all measurable sets $A_1, ..., A_n$,

$$\Pr(X_1 \in A_1, ..., X_n \in A_n) = \int_{\Delta(S)} Q(A_1) \cdots Q(A_n) \mu(dQ).$$

[18] One can assume that there are an infinite number of potential players in the auction, but for some reason only a finite number of them actually participate. Then, one can apply de Finetti's theorem. However, this will be of course *with* a loss of generality.

[19] Example 5.2 generalizes an example given by Diaconis and Freedman (1980). They prove an approximation version of de Finetti's theorem for a finite set of random variables.

this, assume that the pdf of the signals conditional to v, $f(t_1, ..., t_n|v)$, is C^2 (twice continuously differentiable) and has full support. It can be proven that the signals are affiliated if

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial t_j} \geqslant 0,$$

and

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial v} \geqslant 0, \tag{2}$$

for all i, j (see Topkis (1978, p. 310)). It is important to note that conditional independence implies only that

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial t_j} = 0.$$

Thus, conditional independence is not sufficient for affiliation. To obtain affiliation, one needs to assume (2) above, i.e., that t_i and v are affiliated. In other words, to obtain affiliation from conditional independence, one has to assume affiliation itself. Thus, conditional independence does not give an economic justification for affiliation.

The fact that we are not able to find a justification in the general model of conditional independence does not imply that it does not exist, at least in special cases.

There is a particular conditional independence model where affiliation can be reasonably justified. Assume that the signals t_i are a common value plus an individual error, that is, $t_i = v + \varepsilon_i$, where the ε_i are independent and identically distributed. Now, we almost have the result that the signals t_1, ..., t_n are affiliated: it is still necessary to assume an additional condition. Let g be the pdf of the ε_i, $i = 1$, ..., n. Then, $t_1, ..., t_n$ are affiliated if and only if g is a strongly unimodal function.[20,21]

[20]The term is borrowed from Lehmann (1986). A function is strongly unimodal if $\log g$ is concave. A proof of the affirmation can be found in Lehmann (1986, Example 1, p. 509), or obtained directly from the previous discussion.

[21]Even if g is strongly unimodal, so that $t_1, ..., t_n$ are affiliated, it is not true in general that t_1, ..., t_n, ε_1, ..., ε_n, v are affiliated.

5.3 The use of affiliation in other sciences

As we commented in the introduction, affiliation is used—under other names—in other sciences. Thus, a natural question would be: "Does the justification for affiliation in other sciences carry over into economics?"

Affiliation is used in statistics, as Positive Likelihood Ratio Dependence, the name given by Lehmann (1966) when he introduced the concept, or in reliability theory, as Total Positivity of order 2 (TP_2) for the case of two variables, or Multivariate Total Positivity of Order 2 (MTP_2) for n variables, after Karlin (1968). TP_2 is used when there are good reasons for adopting special distributions in some problems, and those distributions happen to satisfy the TP_2 condition. An example of this can be seen in the historical notes of Barlow and Proschan (1965, Chapter 1) about reliability theory. It is natural to assume that the failure rates of components or systems follow specific probabilistic distributions (exponentials, for instance), and such special distributions have the TP_2 property. Thus, the corresponding theory of total positive distributions can be advantageously used. Another example of this is the use of copulas.[22] If we assume that the distribution is in a family of copulas that have the MTP property, then the use of affiliation's properties and implications is advantageous and justified by the choice of the set of distribution functions.

In the case of economic models, especially auction theory, the random variables (types) represent information gathered by the bidders. There are some situations where we can assume special forms of distributions, but in general there is no justification for such assumptions. In fact, specific distributions are rarely assumed in the theory.[23] Thus, the compelling justification that is presented for applications in reliability theory or statistics does not seem appropriate in economic settings.

6 Related literature

Few papers have pointed out restrictions or limitations to the implications of affiliation. Perry and Reny (1999) presented an example of a multi-unit auction where the linkage principle fails and the revenue ranking is reversed, even under affiliation. This result shows that revenue ranking is not robust when the number

[22] See, for instance, Li, Paarsch, and Hubbard (2007).

[23] McAfee and Vincent (1992) make a similar observation, when they note the "lack of any a priori guidance about the appropriate distribution" (p. 512).

of objects increases from one to many. In contrast, one of our results shows that the revenue ranking is not robust even if we maintain the number of objects but allow for other kinds of dependences. Klemperer (2003) argues that, in real auctions, affiliation is not as important as asymmetry and collusion and illustrates his arguments with examples of the 3G auctions conducted in Europe in 2000–2001.

Nevertheless, much more has been written in accordance with the conclusions of affiliation. McMillan (1994, p.152) says that the auction theorists working as consultants to the FCC in spectrum auctions advocated for the adoption of the open auction using the linkage principle as one of the arguments: "Theory says, then, that the government can increase its revenue by publicizing any available information that affects the licensee's assessed value."[24] Milgrom (1989) emphasizes affiliation as the explanation for the predominance of the English auction over the first-price auction.

On the other hand, the experimental and empirical literature show an amazing lack of studies about whether affiliation holds or not. The empirical literature has tested affiliation's implication that the English auction gives higher revenue than the first-price auction, but there is no clear confirmation of this prediction. See Laffont (1997) for a survey of empirical literature on auctions. We are aware of only three papers proposing tests of affiliation: de Castro and Paarsch (2010), Jun, Pinkse, and Wan (2010) and Li and Zhang (2010). Those papers were motivated by an earlier version of this paper. The available experimental studies investigated only some of the implications of affiliation. See Kagel (1995) for a survey of this literature. See also section 7 below for suggestions of future work regarding this topic.

7 Conclusion: the need of new studies

As we observed in the introduction, there is no question that dependence is of fundamental importance in economics. It is also clear that we have experienced an astonishing progress since affiliation was introduced as a foundation for the study of dependence by Milgrom and Weber (1982a). Almost thirty years later, a critical reassessment of the assumption seems overdue.

This paper shows that affiliation imposes some restrictions that have not been investigated in detail. The intuitive appeal of affiliation is clear, yet as demon-

[24]Note that this is not necessarily their main argument, since they mentioned other advantages of the open auction, as "the bidders' ability to learn from other bids in the auction." McMillan (1994, p.152)

strated in this paper, there are other ways to describe positive dependence that are no less intuitive, but that have very different implications.

Although we briefly reviewed the experimental and empirical literature, the scope of this paper was mainly theoretical. As we have seen, the respective literatures miss comprehensive studies about this topic.

Experimental studies could shed light on the actual distribution of values across individuals, controlling for the common knowledge.[25] It would be very helpful to develop methods to determine the values that people attribute to objects in an auction and whether those values are correlated or not. With respect to econometrics, an obvious need is to develop methods to test the affiliation of bidders' values, controlling for the common knowledge (if this is possible). It would also be useful to develop techniques to describe the kind of dependence of the bids in real auctions. It would be very helpful to learn whether the kind of dependence is different across different markets and how these differences can be characterized. For instance, is there less correlation in Internet auctions, where the participants are consumers with almost no interaction, than in auctions where the participants are firms or professionals acting in the same industry? Yet another direction of research would be the development of econometric techniques to deal with dependence out of affiliation.[26]

It should be noted that the assessment presented in this paper is not a criticism of Milgrom and Weber (1982a)'s important results. On the contrary, Theorems 4.1 and 4.2 can be interpreted as saying that they have found not only a sufficient condition for their results, but also practically the most general one.[27]

On the other hand, this paper tries to deepen our understanding of affiliation, and how it relates to other aspects of positive dependence. Our results suggest that substantive progress in this field should require new approaches to dependence in economics.[28]

[25]The importance of controlling for common knowledge is further discussed in de Castro (2008). Experiments have an advantage in this aspect, because they can control for "unobserved heterogeneity" that econometricians cannot.

[26]Grid distributions can be useful for this task. See de Castro and Paarsch (2010).

[27]Although we generalize their results to property VI in the particular case of private values, this property is yet close to affiliation (property VII).

[28] In this direction, de Castro (2008) proposes the use of grid distributions to study not only dependence but also asymmetric priors in games of incomplete information.

A Proofs

A.1 Proof of Theorem 5.1 .

First, we prove that $\mathcal{C}\backslash\mathcal{A}$ is open. If $f \in \mathcal{C}\backslash\mathcal{A}$, then

$$f(x) f(x') > f(x \wedge x') f(x \vee x'),$$

for some $x, x' \in [0,1]^n$. Fix such x and x' and define $K = f(x) + f(x') + f(x \wedge x') + f(x \vee x') > 0$. Choose $\varepsilon > 0$ such that $2\varepsilon K < f(x) f(x') - f(x \wedge x') f(x \vee x')$ and let $B_\varepsilon(f)$ be the open ball with radius ε and center in f. Thus, if $g \in B_\varepsilon(f)$, $\|f-g\| < \varepsilon$, which implies $g(x) > f(x) - \varepsilon$, $g(x') > f(x') - \varepsilon$, $g(x \wedge x') < f(x \wedge x') + \varepsilon$, $g(x \vee x') < f(x \vee x') + \varepsilon$, so that

$$\begin{aligned}
& g(x) g(x') - g(x \wedge x') g(x \vee x') \\
> \; & [f(x) - \varepsilon][f(x') - \varepsilon] - [f(x \wedge x') + \varepsilon][f(x \vee x') + \varepsilon] \\
= \; & f(x) f(x') - f(x \wedge x') f(x \vee x') - \varepsilon [f(x) + f(x') + f(x \wedge x') + f(x \vee x')] \\
= \; & f(x) f(x') - f(x \wedge x') f(x \vee x') - \varepsilon K \\
> \; & \varepsilon K > 0,
\end{aligned}$$

which implies that $B_\varepsilon(f) \subset \mathcal{C}\backslash\mathcal{A}$, as we wanted to show.

Now, let us show that $\mathcal{C}\backslash\mathcal{A}$ is dense, that is, given $f \in \mathcal{C}$ and $\varepsilon > 0$, there exists $g \in B_\varepsilon(f) \cap \mathcal{C}\backslash\mathcal{A}$. Since $f \in \mathcal{C}$, it is uniformly continuous (because $[0,1]^n$ is compact), that is, given $\eta > 0$, there exists $\delta > 0$ such that $\|x - x'\|_{\mathbb{R}^n} < 2\delta$ implies $|f(x) - f(x')| < \eta$. Take $\eta = \varepsilon/4$ and the corresponding δ.

Choose $a \in (4\delta, 1 - 4\delta)$ and define x (x') by specifying that their first $\left\lfloor \frac{n}{2} \right\rfloor$ coordinates are equal to $a - \delta$ $(a + \delta)$ and the last ones to be equal to $a + \delta$ $(a - \delta)$. Thus, $x \wedge x' = (a - \delta, ..., a - \delta)$ and $x \vee x' = (a + \delta, ..., a + \delta)$. Let x_0 denote the vector $(a, ..., a)$. For $y = x$, x', $x \wedge x'$ or $x \vee x'$, we have: $|f(y) - f(x_0)| < \eta$. Let $\xi : (-1,1)^n \to \mathbb{R}$ be a smooth function that vanishes outside $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)^n$ and equals 1 in $\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^n$. Define the function g by

$$\begin{aligned}
g(y) \;=\; & f(y) + 2\eta\xi(y - x) + 2\eta\xi(y - x') \\
& -2\eta\xi(y - x \wedge x') - 2\eta\xi(y - x \vee x').
\end{aligned}$$

Observe that $\|g - f\| = 2\eta = \varepsilon/2$, that is, $g \in B_\varepsilon(f)$. In fact, $g \in B_\varepsilon(f) \cap \mathcal{C}\backslash A$, because

$$\begin{aligned} g(x) &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x') &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x \wedge x') &= f(x \wedge x') - 2\eta < f(x_0) - \eta; \\ g(x \vee x') &= f(x \vee x') - 2\eta < f(x_0) - \eta, \end{aligned}$$

which implies

$$\begin{aligned} & g(x)\, g(x') - g(x \wedge x')\, g(x \vee x') \\ > \; & [f(x_0) + \eta]^2 - [f(x_0) - \eta]^2 \\ = \; & 4\eta > 0, \end{aligned}$$

as we wanted to show. ▮

A.2 Proof of Theorem 3.1.

The proof of Theorem 3.1 is divided in two parts: the implications and the counterexamples.

A.2.1 Implications

It is obvious that $(III) \Rightarrow (II) \Rightarrow (I)$. The implication $(IV) \Rightarrow (III)$ is Theorem 4.3. of Esary, Proschan, and Walkup (1967). The implication $(V) \Rightarrow (IV)$ is proved by Tong (1980, p. 80). The implication $(VII) \Rightarrow (VI)$ is Lemma 1 of Milgrom and Weber (1982a). Thus, we need only to prove $(VI) \Rightarrow (V)$.

For this, assume that $H(y|x) \equiv \frac{f(y|x)}{F(y|x)}$ is non-decreasing in x for all y. Then, $H(y|x) = \partial_y [\ln F(y|x)]$ and we have

$$1 - \ln[F(y|x)] = \int_y^\infty H(s|x)\, ds \geqslant \int_y^\infty H(s|x')\, ds = 1 - \ln[F(y|x')],$$

if $x \geqslant x'$. Then, $\ln[F(y|x)] \leqslant \ln[F(y|x')]$, which implies that $F(y|x)$ is non-increasing in x for all y, as required by Property V.

A.2.2 Counterexamples

The counterexamples for each passage are given by Tong (1980, Chapter 5), except those involving Property (VI): $(V) \nRightarrow (VI)$, $(VI) \nRightarrow (VII)$. For the counterexample of $(V) \nRightarrow (VI)$, consider the following symmetric and continuous pdf defined on $[0,1]^2$:

$$f(x,y) = \frac{d}{1+4(y-x)^2}$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$ is the suitable constant for f to be a pdf. We have the marginal given by

$$f(y) = \frac{d}{2}\left[\arctan 2(1-y) + \arctan 2(y)\right]$$

so that we have, for $(x,y) \in [0,1]^2$:

$$f(x|y) = 2\left[1+4(y-x)^2\right]^{-1}\left[\arctan 2(1-y) + \arctan 2(y)\right]^{-1},$$

$$F(x|y) = \frac{\left[\arctan 2(x-y) + \arctan 2(y)\right]}{\arctan 2(1-y) + \arctan 2(y)}$$

and

$$\frac{F(x|y)}{f(x|y)} = \frac{1}{2}\left[1+4(y-x)^2\right]\left[\arctan(2x-2y) + \arctan(2y)\right].$$

Observe that for $y' = 0.91 > y = 0.9$ and $x = 0.1$,

$$\frac{F(x|y')}{f(x|y')} > \frac{F(x|y)}{f(x|y)},$$

which violates Property (VI). On the other hand,

$$\begin{aligned}\partial_y\left[F(x|y)\right] &= \frac{\frac{2}{1+4y^2} - \frac{2}{1+4(x-y)^2}}{\arctan(2-2y) + \arctan(2y)} \\ &\quad - \frac{\left[\arctan(2x-2y) + \arctan(2y)\right]\left[\frac{2}{1+4y^2} - \frac{2}{1+4(1-y)^2}\right]}{\left[\arctan(2-2y) + \arctan(2y)\right]^2}\end{aligned}$$

In the considered range, the above expression is non-positive, so that Property (V) is satisfied. Then, $(V) \nRightarrow (VI)$.

Now we will establish that $(VI) \nRightarrow (VII)$. Fix an $\varepsilon < 1/2$ and consider the symmetric density function over $[0,1]^2$:

$$f(x,y) = \begin{cases} k_1, & \text{if } x + y \leqslant 2 - \varepsilon \\ k_2, & \text{otherwise} \end{cases}$$

where $k_1 > 1 > k_2 = 2\left[1 - k_1\left(1 - \varepsilon^2/2\right)\right]/\varepsilon^2 > 0$ and $\varepsilon \in (0, 1/2)$. For instance, we could choose $\varepsilon = 1/3$, $k_1 = 19/18$ and $k_2 = 1/18$. The conditional density function is given by

$$f(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and the conditional c.d.f. is given by:

$$F(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1 y}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2(y+x+\varepsilon-2)+k_1(2-\varepsilon-x)}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and

$$\frac{F(y|x)}{f(y|x)} = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ y, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ y + x + \varepsilon - 2 + k_1/k_2\left(2 - \varepsilon - x\right), & \text{otherwise} \end{cases}$$

Since $1 - k_1/k_2 < 0$, the above expression is non-increasing in x for all y, so that Property (VI) is satisfied. On the other hand, it is obvious that Property (VII) does not hold:

$$f(0.5, 0.5)\, f\left(1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right) = k_2 k_1 < k_1^2 = f\left(0.5, 1 - \frac{\varepsilon}{2}\right) f\left(0.5, 1 - \frac{\varepsilon}{2}\right).$$

This shows that $(VI) \nRightarrow (VII)$.

A.3 Proof of Theorem 4.1.

The equilibrium existence follows from Milgrom and Weber (1982a)'s proof. The counterexample is in continuous values, but using the grid distributions proposed by de Castro (2008).[29] Consider the grid distribution $f : [0,1]^2 \to \mathbb{R}_+, f \in \mathcal{D}^4$ defined by:

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right],$$

for $m, p \in \{1,2,3,4\}$, where

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 2.0797 & 0.5505 & 1.4000 & 0.2296 \\ 0.5505 & 0.6965 & 0.5504 & 0.2439 \\ 1.4000 & 0.5504 & 2.3395 & 1.8158 \\ 0.2296 & 0.2439 & 1.8158 & 1.3040 \end{bmatrix}.$$

The definition of f at the zero measure set of points $\{(x,y) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0$ or $p = 0\}$ is arbitrary. This distribution satisfies Property V but there does not exist a symmetric monotonic pure strategy equilibrium. These claims can be verified directly through tedious and lengthy calculations available upon request.

A.4 Proof of Theorem 4.2.

The dominant strategy for each bidder in the second-price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second-price auction, P^2, is given by:

$$\begin{aligned} P^2 &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} y f(y|x)\, dy \cdot f(x)\, dx = \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} [y - b(y)]\, f(y|x)\, dy \cdot f(x)\, dx + \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\, dy \cdot f(x)\, dx, \end{aligned}$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first-price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx$,

[29] I was unable to find examples with continuous variables without using grid distributions. At some point, I believed that I have found one non-grid distribution example, but Robert Wilson pointed out an inconsistency to me. I am grateful to him for this.

from the first-order condition: $b^{'}(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$\begin{aligned}
& \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y) f(y|x) \, dy \cdot f(x) \, dx \\
= & \int_{[\underline{t},\bar{t}]} \left[b(x) F(x|x) - \int_{[\underline{t},x]} b^{'}(y) F(y|x) \, dy \right] \cdot f(x) \, dx \\
= & \int_{[\underline{t},\bar{t}]} b(x) F(x|x) \cdot f(x) \, dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x) \, dy \cdot f(x) \, dx
\end{aligned}$$

In the last line, the first integral is just the expected payment for the first-price auction, P^1. Thus, we have

$$\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x) \, dy \cdot f(x) \, dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x) \, dy \cdot f(x) \, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x) \, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x) \, dy \cdot f(x) \, dx
\end{aligned}$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t) \, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$\begin{aligned}
b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y) \, d\alpha \\
&= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y) \, d\alpha.
\end{aligned}$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx,
\end{aligned}
$$

which is the desired expression if we multiply by the number n of players.

For the counterexample, consider the grid distribution $f : [0,1]^2 \to \mathbb{R}_+, f \in \mathcal{D}^4$ defined in the same fashion as in the proof of Theorem 4.1, by:

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 2.7468 & 0.0803 & 0.1195 & 0.0696 \\ 0.0803 & 0.3200 & 0.5271 & 0.1224 \\ 0.1195 & 0.5271 & 1.7814 & 0.5650 \\ 0.0696 & 0.1224 & 0.5650 & 1.2705 \end{bmatrix}.
$$

This distribution satisfies Property V (but not Property VI). Moreover, the first-price auction with this distribution has a SMPSE and a higher expected revenue than the correspondent second-price auction ($R_2 < R_1$). Again, these claims can be verified directly through tedious calculations.

A.5 Proof of Example 5.2.

Let p denote the probability of Heads and let μ be a distribution over coins. Then: $\Pr(\text{Heads,Heads}) = \varepsilon = \int (p)^2 \mu(dp)$, and $\Pr(\text{Tails,Tails}) = \varepsilon = \int (1 - p)^2 \mu(dp) = \int (1)\mu(dp) + \int (-2p)\mu(dp) + \int (p)^2 \mu(dp) = 1 - 2E[p] + \varepsilon$. Then, $1 - 2E[p] = 0$, or $E[p] = 1/2$. This implies: Var[p]= $\int (p - E[p])^2 \mu(dp) = \int (p^2 - p + \frac{1}{4})\mu(dp) = \int (p)^2 \mu(dp) - \frac{1}{4} = \varepsilon - \frac{1}{4}$. Since Var[p] is non-negative, $\varepsilon \geqslant \frac{1}{4}$. ∎

References

ATHEY, S. (2001): "Single Crossing Properties and the Existence of Prue Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69(4), 861–99.

AUMANN, R. J. (1976): "Agreeing to Disagree," *The Annals of Statistics*, 4(6), 1236–1239.

BARLOW, R. E., AND F. PROSCHAN (1965): *Mathematical theory of reliability*. John Wiley & Sons.

BERGIN, J. (2001): "Common Knowledge with Monotone Statistics," *Econometrica*, 69(5), 1315–1332.

DE CASTRO, L. (2007): "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions," Discussion paper, Working paper, Universidad Carlos III de Madrid.

——— (2008): "Grid Distribution to Study Single Obejct Auctions," Discussion paper, University of Illinois at Urbana-Champaign.

DE CASTRO, L. I., AND H. J. PAARSCH (2010): "Testing Affiliation in Private-Values Models of First-Price Auctions Using Grid Distributions," *The Annals of Applied Statistics*, 4(4), 2073–2098.

DIACONIS, P., AND D. FREEDMAN (1980): "Finite exchangeable sequences," *Ann. Probab.*, 8(4), 745–764.

ESARY, J., F. PROSCHAN, AND D. WALKUP (1967): "Association of random variables, with applications," *The Annals of Mathematical Statistics*, 38, 1466–1474.

HEWITT, E., AND L. J. SAVAGE (1955): "Symmetric measures on Cartesian products," *Trans. Amer. Math. Soc.*, 80, 470–501.

JACKSON, M. O., AND J. M. SWINKELS (2005): "Existence of Equilibrium in Single and Double Private Value Auctions," *Econometrica*, 73(1), 93–139.

JUN, S., J. PINKSE, AND Y. WAN (2010): "A consistent nonparametric test of affiliation in auction models," *Journal of Econometrics*, 159, 46–54.

KAGEL, J. H. (1995): "Auctions: A Survey of Experimental Research," *In: Kagel, J. H. and Roth, A., The Handbook of Experimental Economics, Princeton University Press,*, pp. 501–85.

KARLIN, S. (1968): *Total positivity*. Stanford University Press, Stanford, Calif.

KLEMPERER, P. (2003): "Alfred Marshall Lecture: Using and Abusing Economic Theory," *Journal of the European Economic Association*, 1(2-3), 272–300.

KLEMPERER, P. (2004): *Auctions: Theory and Practice*. Princeton University Press.

LAFFONT, J. (1997): "Game theory and empirical economics: The case of auction data," *European Economic Review*, 41(1), 1–35.

LEHMANN, E. (1966): "Some concepts of dependence," *Annals of Mathematical Statistics*, 37, 1137–1153.

——— (1986): *Testing statistical hypotheses*. John Wiley & Sons, 2nd edn.

LI, T., H. PAARSCH, AND T. HUBBARD (2007): "Semiparametric Estimation in Models of First-Price, Sealed-Bid Auctions with Affiliation," Discussion paper, working paper.

LI, T., AND B. ZHANG (2010): "Testing for Affiliation in First-Price Auctions Using Entry Behavior ," *International Economic Review*, 51(3), 837–850.

MASKIN, E., AND J. RILEY (1984): "Optimal Auctions with Risk Averse Buyers," *Econometrica*, 52(6), 1473–1518.

MCAFEE, R., AND D. VINCENT (1992): "Updating the Reserve Price in Common Value Auctions," *American Economic Review, Papers and Proceedings*, 82(2), 512–518.

MCMILLAN, J. (1994): "Selling Spectrum Rights," *Journal of Economic Perspectives*, 8(3), 145–162.

MILGROM, P. (1981a): "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12(2), 380–391.

——— (1981b): "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica*, 49(4), 921–43.

——— (1989): "Auctions and Bidding: A Primer," *Journal of Economic Perspectives*, 3(3), 3–22.

MILGROM, P., AND R. WEBER (1982a): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50(5), 1089–1122.

MILGROM, P., AND R. J. WEBER (1982b): "The value of information in a sealed-bid auction," *Journal of Mathematical Economics*, 10(1), 105 – 114.

MONTEIRO, P., AND H. MOREIRA (2006): "First-price auctions without affiliation," *Economics Letters*, 91(1), 1–7.

PERRY, M., AND P. RENY (1999): "On the Failure of the Linkage Principle in Multi-unit Auctions," *Econometrica*, 67(4), 895–900.

PERSICO, N. (2000): "Information Acquisition in Auctions," *Econometrica*, 68(1), 135–148.

RILEY, J. G. (1988): "Ex Post Information in Auctions," *The Review of Economic Studies*, 55(3), 409–429.

SOBEL, J. (2006): "Information Aggregation and Group Decisions," *Unpublished Paper*.

TONG, Y. (1980): "Probability inequalities in multivariate distributions.," *New York: Academic Press, 1980.*

TOPKIS, D. (1978): "Minimizing a submodular function on a lattice," *Operations Research*, 26, 305–321.

VAN ZANDT, T., AND X. VIVES (2007): "Monotone equilibria in Bayesian games of strategic complementarities," *Journal of Economic Theory*, 134(1), 339 – 360.

WILSON, R. (1969): "Competitive bidding with disparate information," *Management Science*, 15, 446–448.

WILSON, R. (1977): "A bidding model of perfect competition," *The Review of Economic Studies*, 44(3), 511–518.

YANAGIMOTO, T. (1972): "Families of positively dependent random variables," *Annals of the Institute of Statistical Mathematics*, 24(1), 559–573.