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Production Networks: A Primer
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Online Appendix for "Production Networks: A Primer"

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This appendix contains the proofs and derivations omitted from the main body of the paper. Section A derives Equation (11) in the paper. Section B provides the proof of Theorem 4.

A CES Production Technologies

In what follows, we derive the expression in Equation (11) in the paper. Suppose that the production technology of firms in industry i is given by Equation (10) in the paper. The first-order conditions of firms in industry i are therefore given by

$$l_i = \alpha_i p_i y_i / w \tag{A.1}$$

$$x_{ij} = (1-\alpha_i) a_{ij} p_i y_i p_j{}^{-\sigma_i} \left(\sum_{k=1}^{n} a_{ik} p_k^{1-\sigma_i} \right)^{-1}, \tag{A.2}$$

where we are using the fact that $\alpha_i + \sum_{j=1}^n a_{ij} = 1$ for all i. Plugging the above expressions back into the production function of firms in industry i implies that

$$p_i z_i = w^{\alpha_i} \left(\frac{1}{1-\alpha_i} \sum_{k=1}^{n} a_{ik} p_k^{1-\sigma_i} \right)^{(1-\alpha_i)/(1-\sigma_i)}.$$

Taking logarithms from both sides of the above equation leads to the following system of equations

$$\log(p_i/w) = -\epsilon_i + \frac{1-\alpha_i}{\sigma_i - 1} \log \left(\frac{1}{1-\alpha_i} \sum_{k=1}^{n} a_{ik} (p_k/w)^{1-\sigma_i} \right).$$

We make two observations. First, the above system of equations immediately implies that when $\epsilon_i = 0$ for all industries i, then all relative prices coincide with another, that is, $p_i = w$ for all i. Second, differentiating both sides of the above equation with respect to ϵ_j and evaluating it at $\epsilon = 0$ leads to $d\hat{p}_i/d\epsilon_j = -\mathbb{I}_{\{i=j\}} + \sum_{k=1}^n a_{ik} d\hat{p}_k/\epsilon_j$, where recall that $\hat{p}_i = \log(p_i/w)$ is the log relative price of good i and $\mathbb{I}$ denotes the indicator function. Rewriting the previous equation in matrix form, we obtain $d\hat{p}/d\epsilon_j = -e_j + \mathbf{A} d\hat{p}/d\epsilon_j$, where e_j is the j-th unit vector. Consequently, $d\hat{p}/d\epsilon_j = -(\mathbf{I} - \mathbf{A})^{-1} e_j$, which in turn can be rewritten as

$$\left. \frac{d\hat{p}_i}{d\epsilon_j} \right|_{\epsilon=0} = -\ell_{ij}. \tag{A.3}$$

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The above equation therefore illustrates how shocks to industry j change the relative prices of all other industries up to a first-order approximation.

Next, recall that the market-clearing condition for good i is given by $y_i = c_i + \sum_{j=1}^n x_{ji}$. Multiplying both sides by p_i and dividing by GDP implies that

$$\lambda_i = \beta_i + \sum_{k=1}^n \omega_{ki}\lambda_k,$$

where $\lambda_i = p_i y_i/\,\mathrm{GDP}$ is the Domar weight of industry i and $\omega_{ki} = p_i x_{ki}/p_k y_k$. Note that in deriving the above equation, we are using the fact that the household's first-order condition requires that $p_i c_i = \beta_i\,\mathrm{GDP}$. Differentiating both sides of the above equation with respect to ϵ_j implies that

$$\frac{d\lambda_i}{d\epsilon_j} = \sum_{k=1}^n \omega_{ki}\frac{d\lambda_k}{d\epsilon_j} + \sum_{k=1}^n \lambda_k \frac{d\omega_{ki}}{d\epsilon_j}. \tag{A.4}$$

On the other hand, Equation (A.2) implies that $\omega_{ki} = (1-\alpha_k)a_{ki}p_i^{1-\sigma_k}/\left(\sum_{r=1}^n a_{kr}p_r^{1-\sigma_k}\right)$. Hence, differentiating both sides of this expression, evaluating them at $\epsilon = 0$, and plugging the resulting expression back into Equation (A.4) implies that

$$\frac{d\lambda_i}{d\epsilon_j} = \sum_{k=1}^n a_{ki}\frac{d\lambda_k}{d\epsilon_j} + \sum_{k=1}^n (1-\sigma_k)a_{ki}\lambda_k\left(\frac{d\hat{p}_i}{d\epsilon_j} - \frac{1}{1-\alpha_k}\sum_{r=1}^n a_{kr}\frac{d\hat{p}_r}{d\epsilon_j}\right).$$

Hence, using Equation (A.3), we obtain

$$\frac{d\lambda_i}{d\epsilon_j} - \sum_{k=1}^n a_{ki}\frac{d\lambda_k}{d\epsilon_j} = \sum_{k=1}^n (\sigma_k - 1)a_{ki}\lambda_k\left(\ell_{ij} - \frac{1}{1-\alpha_k}\sum_{r=1}^n a_{kr}\ell_{rj}\right).$$

Multiplying both sides of the above equation by ℓ_{is}, summing over all i, and noting that $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1}$ leads to

$$\frac{d\lambda_i}{d\epsilon_j} = \sum_{k=1}^n (\sigma_k - 1)\lambda_k\left(\sum_{s=1}^n a_{ks}\ell_{si}\ell_{sj} - \frac{1}{1-\alpha_k}\sum_{r=1}^n a_{kr}\ell_{rj}\sum_{s=1}^n a_{ks}\ell_{si}\right). \tag{A.5}$$

On the other hand, the fact that $\lambda_i = p_i y_i/\,\mathrm{GDP}$ implies that

$$\frac{d\log y_i}{d\epsilon_j} = -\frac{d\hat{p}_i}{d\epsilon_j} + \frac{1}{\lambda_i}\frac{d\lambda_i}{d\epsilon_j} = \ell_{ij} + \frac{1}{\lambda_i}\frac{d\lambda_i}{d\epsilon_j},$$

where the second equality is a consequence of Equation (A.3). Plugging for $d\lambda_i/d\epsilon_j$ from Equation (A.5) into the above equation leads to Equation (11). □

B Proof of Theorem 4

Consider two economies with symmetric circulant input-output matrices $\mathbf{A}$ and $\tilde{\mathbf{A}}$ and suppose the latter is more interconnected than the former, that is, there exists a $\gamma \in [0,1]$ such that

$$\tilde{\mathbf{A}} = \gamma\mathbf{A} + (1-\gamma)(1-\alpha)\mathbf{J},$$

where $\mathbf{J} = (1/n)\mathbf{1}\mathbf{1}'$ is a matrix with all entries equal to $1/n$. We first prove statement (b) of the theorem by showing that the above transformation can only decrease the volatility of each industry, i.e., $\mathrm{var}(\log\tilde{y}_i) \leq \mathrm{var}(\log y_i)$ for all i. We then use this result to establish statement (a).

Proof of statement (b). Recall from Theorem 1 that the output of industry i satisfies $\log y_i = \sum_{i=1}^n \ell_{ij}\epsilon_j$. Under our assumption that all microeconomic shocks are i.i.d. with a common variance $\sigma^2 < \infty$, it is immediate that $\mathrm{var}(\log y_i) = \sigma^2 \sum_{j=1}^n \ell_{ij}^2$. Therefore, sectoral log outputs are more volatile in the less interconnected economy (that is, $\mathrm{var}(\log \tilde{y}_i) \leq \mathrm{var}(\log y_i)$ for all i) if and only if $\sum_{j=1}^n \tilde{\ell}_{ij}^2 \leq \sum_{j=1}^n \ell_{ij}^2$ for all i. On the other hand, the assumption that input-output matrices $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are symmetric and circulant implies that $\sum_{j=1}^n \tilde{\ell}_{ij}^2 = (1/n)\sum_{i,j=1}^n \tilde{\ell}_{ij}^2 = (1/n)\,\mathrm{trace}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}}) = (1/n)\,\mathrm{trace}(\tilde{\mathbf{L}}^2)$. Hence, it is sufficient to show that

$$\frac{d}{d\gamma}\,\mathrm{trace}(\tilde{\mathbf{L}}^2)\Big|_{\gamma=1} \geq 0. \tag{B.1}$$

To this end, first note that, by definition, $\tilde{\mathbf{L}} = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}$. Therefore, differentiating $\tilde{\mathbf{L}}^2$ with respect to γ leads to

$$d\tilde{\mathbf{L}}^2/d\gamma = \tilde{\mathbf{L}}^2(d\tilde{\mathbf{A}}/d\gamma)\tilde{\mathbf{L}} + \tilde{\mathbf{L}}(d\tilde{\mathbf{A}}/d\gamma)\tilde{\mathbf{L}}^2.$$

On the other hand, $d\tilde{\mathbf{A}}/d\gamma = \mathbf{A} - (1-\alpha)\mathbf{J}$. Consequently,

$$\begin{aligned}\frac{d\tilde{\mathbf{L}}^2}{d\gamma}\Big|_{\gamma=1} &= \mathbf{L}^2\mathbf{A}\mathbf{L} + \mathbf{L}\mathbf{A}\mathbf{L}^2 - (1-\alpha)(\mathbf{L}^2\mathbf{J}\mathbf{L} + \mathbf{L}\mathbf{J}\mathbf{L}^2) \\ &= 2(\mathbf{L}^3 - \mathbf{L}^2) - 2(1-\alpha)\alpha^{-3}\mathbf{J},\end{aligned}$$

where the second equality uses $\mathbf{L}\mathbf{A} = \mathbf{A}\mathbf{L} = \mathbf{L} - \mathbf{I}$ and the fact that the row and column sums of $\mathbf{L}$ are equal to $1/\alpha$, i.e., $\mathbf{L}\mathbf{1} = \mathbf{L}'\mathbf{1} = (1/\alpha)\mathbf{1}$. Hence,

$$\frac{d}{d\gamma}\,\mathrm{trace}(\tilde{\mathbf{L}}^2)\Big|_{\gamma=1} = 2\,\mathrm{trace}(\mathbf{L}^3) - 2\,\mathrm{trace}(\mathbf{L}^2) - 2(1-\alpha)/\alpha^3.$$

Note that the trace of a matrix is equal to the sum of its eigenvalues. Furthermore, the fact that $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ implies that $\lambda_k(\mathbf{L}) = (1-\lambda_k(\mathbf{A}))^{-1}$, where $\lambda_k(\mathbf{L})$ and $\lambda_k(\mathbf{A})$ are the k-th largest eigenvalues of $\mathbf{L}$ and $\mathbf{A}$, respectively. Consequently,

$$\frac{d}{d\gamma}\,\mathrm{trace}(\tilde{\mathbf{L}}^2)\Big|_{\gamma=1} = 2\sum_{k=1}^n \frac{1}{(1-\lambda_k(\mathbf{A}))^3} - 2\sum_{k=1}^n \frac{1}{(1-\lambda_k(\mathbf{A}))^2} - 2(1-\alpha)/\alpha^3 = 2\sum_{k=2}^n \frac{\lambda_k(\mathbf{A})}{(1-\lambda_k(\mathbf{A}))^3}.$$

The second equality above is a consequence of the fact that the row sums of matrix $\mathbf{A}$ are all equal to $1-\alpha$, and hence, by the Perron-Frobenius theorem, its largest eigenvalue is given by $\lambda_1(\mathbf{A}) = 1-\alpha$. Multiplying and dividing the right-hand side of the above equation by $n-1$ and using the fact that the function $g(z) = z/(1-z)^3$ is convex over the interval $(-1, 1)$ implies that

$$\frac{d}{d\gamma}\,\mathrm{trace}(\tilde{\mathbf{L}}^2)\Big|_{\gamma=1} \geq \frac{2\sum_{k=2}^n \lambda_k(\mathbf{A})}{(1 - \frac{1}{n-1}\sum_{k=2}^n \lambda_k(\mathbf{A}))^3}. \tag{B.2}$$

Next, note that $\sum_{k=2}^n \lambda_k(\mathbf{A}) = \mathrm{trace}(\mathbf{A}) - \lambda_1(\mathbf{A}) = na_{ii} - (1-\alpha) \geq 0$, where we are using the assumption that $a_{ii} \geq 1/n$ for all i. This implies that the numerator of the fraction on the right-hand side of (B.2) is nonnegative. Furthermore, the fact that $\lambda_k(\mathbf{A}) \leq \lambda_1(\mathbf{A}) = 1-\alpha$ guarantees that the denominator of the fraction on the right-hand side of (B.2) is strictly positive. Taken together, these two observations establish inequality (B.1). □

Proof of statement (a). We now use statement (b) to establish statement (a) of the theorem. Recall from the previous part that the variance-covariance matrix of sectoral log outputs is given by $\sigma^2\tilde{\mathbf{L}}\tilde{\mathbf{L}}'$. On the other hand, the assumption that the input-output matrix $\mathbf{A}$ is symmetric and circulant guarantees that all row and column sums of $\tilde{\mathbf{L}}$ are equal to $1/\alpha$. Therefore,

$$\sum_{i,j=1}^{n} \operatorname{cov}(\log \tilde{y}_i, \log \tilde{y}_j) = \mathbf{1}'\tilde{\mathbf{L}}\tilde{\mathbf{L}}'\mathbf{1} = n/\alpha^2.$$

Furthermore, the assumption that the economy's input-output matrix is circulant implies that all industries are equally volatile, that is, $\operatorname{var}(\log \tilde{y}_i) = \operatorname{var}(\log \tilde{y}_1)$ for all i. Hence,

$$\sum_{i \neq j} \operatorname{cov}(\log \tilde{y}_i, \log \tilde{y}_j) = n(1/\alpha^2 - \operatorname{var}(\log \tilde{y}_1)).$$

Hence, the average pairwise correlation between sectoral log outputs is given by

$$\tilde{\rho} = \frac{1}{n(n-1)} \sum_{i \neq j} \operatorname{corr}(\log \tilde{y}_i, \log \tilde{y}_j) = \frac{1}{(n-1)\operatorname{var}(\log \tilde{y}_1)}(1/\alpha^2 - \operatorname{var}(\log \tilde{y}_1)).$$

Identical derivations for the less interconnected economy with input-output matrix $\mathbf{A}$ imply that

$$\rho = \frac{1}{(n-1)\operatorname{var}(\log y_1)}(1/\alpha^2 - \operatorname{var}(\log y_1)).$$

Comparing the right-hand sides of the above two equations completes the proof: by statement (b) of the theorem, $\operatorname{var}(\log y_1) \geq \operatorname{var}(\log \tilde{y}_1)$, which in turn implies that $\rho \leq \tilde{\rho}$. $\square$